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ABSTRACT

This book is a compilation of five papers presented at conferences between the years 1984 and 1990 on the integration of theory and practice in mathematics education. An introduction acknowledges the chasm felt between theory and practice, and gives an overview of the five papers. The first paper, "Developmental Research on Paper and in Action," deals with the fundamental misunderstanding between teacher and researcher and analyzes their different viewpoints. The second paper, "Ten Major Issues Concerning Systematic Cooperation Between Theory and Practice in Mathematics Education," discusses 10 issues revolving around the problem of teachers being generally skeptical of the benefit of theory and science of mathematics education. The third contribution, "The Dream of Cooperation Between Theory and Practice," discusses the transience of theory. The fourth contribution, "On Levels of Interaction Between Theory and Practice," discusses three levels of interaction between researcher and teacher. The last contribution, "Theory and Practice in Mathematics for the Less Able," epistemologically analyzes the theory-practice problematic. (MDH)

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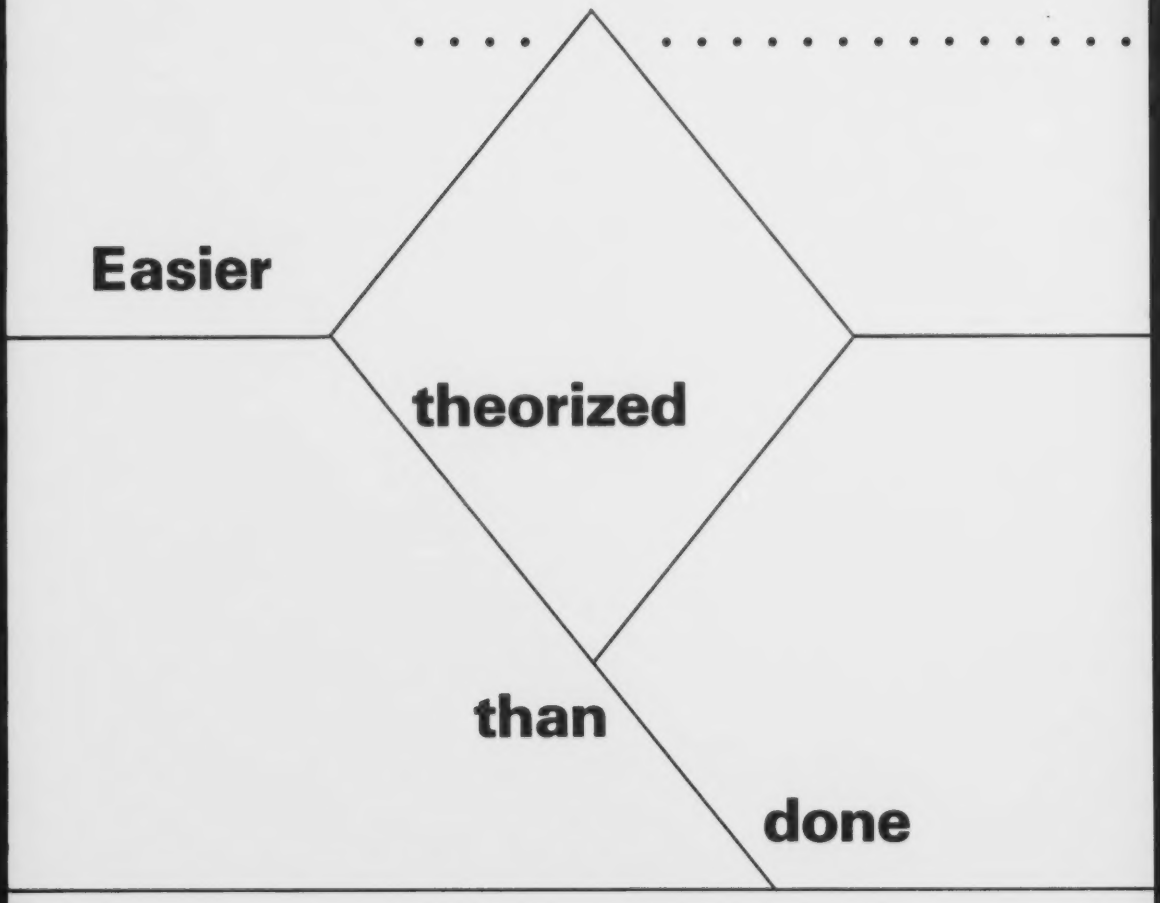

Five papers on the systematic cooperation between theory and practice

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EASIER THEORIZED THAN DONE

Five papers on the Systematic Cooperation between Theory and Practice of mathematics education

Developmental research on paper and in action
Paper presented at the mini-conference on systematic cooperation between theory and practice.
This mini-conference was part of the fifth International Congress on Mathematical Education, Adelaide, Australia, 1984 August 24-30.

Ten major issues concerning systematic cooperation between theory and practice in mathematics education
Paper in preparation to the second conference on systematic co-operation between theory and practice in mathematics education, Lochem, the Netherlands, 1986 November 2 - 7.

The dream of cooperation between theory and practice
Review of a paper presented at the second conference on systematic cooperation between theory and practice in mathematics education, Lochem, the Netherlands, 1986 November 2 - 7.

On levels of interaction between theory and practice
Review and compilation of a paper presented at the third conference on systematic cooperation between theory and practice, London, United Kingdom, Thames Polytechnic, 1988 April 10 - 15 and of an introduction at the sixth International Congress on Mathematical Education, Budapest, Hungary 1988 July 27 - August 3.

Theory and practice in mathematics for the less able
Paper presented at the fourth conference on systematic cooperation between theory and practice in mathematics education, Brakel, Germany 1990 August 17 - 20.

INTRODUCTION

Two main points emerge from the essays that I submitted for the SCTP scheme in 1984-1990. One I have explained as the futility of theory for practice the way the ordinary teacher considers it. It is the doubt over the use of another perspective. The other I have never specifically given, but was always a part of it, and that is my doubt about the use of Mathematics education as pursued in the West. I am greatly astonished about the function and general acceptance of such education, and even about the legitimisation of bad functioning. Should a school only give out "pass" grades or higher, the school as such would no longer be accepted.

In essence a radical sceptical attitude towards the theory as much as the practice is healthy and even necessary, provided it is accompanied by openness. Both belong to the role to be played as either a theoretician or a practicion. If both sides are not open to each other then each relationship, let alone cooperation, is in advance out of the question.

When Bent Christiansen invited me to attend the first conference of 1984 in Adelaide I agreed but inside I felt a little reluctant. What could I contribute to the improvement of the relationship between theory and practice? There has been so much said and written on the subject already. Obviously the problematic occurs in many fields. Think of the interaction between philosophy and psychology, nature and the humanities, and between structuralism and positivism. I set out to speak from my own experience and to translate general solutions to the situation of Mathematics education.

A theory which is limited to an adequate description and prediction of phenomena, a positivistic theory, would be accepted more readily by a practician than an idealistic theory which would want to form reality in such a way as to answer to the ideal. A description is easier than an explanation. An explanation is far more intrusive because it returns to the roots of the thought system. Answering

to an ideal requires changing the foundations. The idealistic and empiricistic viewpoints may be connected complementarily by taking each perception and each thinking as theory-based. In this sense each theory is a world view from a perspective. It is the foundation of the non-monistic philosophy. Whoever does not understand the social-historical development is inconvenienced by the dichotomy between the abstract principle and empirical conceptions.

From the establishment of the "systematic cooperation" it was not intended to view the arena from one meta standpoint and to try to act conciliatorily. Systematic cooperation asked for entanglement, for participation in the antithesis between theory and practice, by acting in relation to the other as theoretician (researcher, developer) or as practician (teacher). Viewed in this light, the terminology SCTP expresses a positivistic attitude so that the systematization approximates an ordering description and does not challenge a search for the ordering principles. Since cooperation demands a tuning of acting as well as a tuning of principles, I have therefore moved in both areas.

From experience it is known that there is a chasm between theory and practice. The paradoxes taught the philosophers that the chasm is of a logical nature and is based on the assumption of the essential difference between knowledge of the world and the world itself, as well as between theory and the application of the theory. As a result, Russell established his hierarchy of types. The essence depends upon the natural independence of the theory. So the chasm is narrowed by declaring theory a cursory independent existence. This means that in the end theory must be dissolved in the changing of the practice. The basic idea of my work is that theory elevates practice by dissolving itself in it. This applies as much for Mathematics education as for Mathematics itself but only when accepting for the latter a relativistic reality.

In my first contribution I dealt with the fundamental misunderstanding between teacher and researcher and I analyzed the different viewpoints. They do not understand each other directly. Practice is

necessary as medium. That is why I suggested that the conversation be directed via a child. The situation is actually even more complex, because theory itself falls apart into several categories when viewed from different perspectives. Especially, the difference between development and research has been described in relation to practice. The method of the curriculum developer with respect to practice is illustrated.

In my second contribution I expanded the query further into around ten sub-problems which all need further elaboration. With this I surpassed the cooperation by a more systematic analysis of the problematic.

In "The dream of cooperation between theory and practice" I discussed the transience of theory. The scepticism expressed in "easier theorized than done" can be conquered in a favourable surrounding. As in theory, so in practice, means that a suitable theory causes its own demise because it becomes swallowed by practice. The best example is the way modern Mathematics has surpassed the pre-industrial, historic Mathematics with the help of a functionalistic perspective. Modern Mathematics reconstructed its positivistic foundation and then incorporated the old Mathematics as a special case, so that positivistic thinking was routed out. As said, the issue is the relativity of reality and the theoretical foundation of the observation. Practice is ever changing, unfortunately too often because of ideology instead of theory.

The fundamental turning point from old to modern Mathematics is analogous to the transition from traditional education to renewed education. The most stringent is the Marxist "negation of the negation", which for the present institutionalised education means being raised to individual learning within the family with the help of modern electronics.

In my fourth contribution I show that the hierarchy idea, as much for the practice of learning as for the teaching of research, is based on the fundamental difference between knowledge and being able. Interesting is its interweaving in the research projects, which show the same phases of development. However, they usually do not get to the functionalistic phase.

In the last contribution the theory-practice problematic is episte-

mologically analyzed. The crux proves to be the ambiguousness of the instruments as has been noted in the first contribution. The epistemological analysis shows that the means can be looked at positivistically as well as structuralistically-theoretically. The designed matters and even the ideas put forward from within curriculum development may therefore be completely misconstrued or misused.

A solution, be it only for weak pupils, offers the functionalistic approach, which is elaborated in the shape of what I call elementary complementaristic education.

The papers remained for years in typescript awaiting a free period in which they could be thoroughly rewritten, with the intention to publish them successively in journals. However, time for this did not come. I was pleasantly surprised to receive requests for my papers. So the decision was made to present these papers together as largely originally written. It is hoped that those who are interested in the theory-practice problem it tackles, will allow for some shortness of explanation, even inelegance of this work.

Enschede, May 1991 Piet Verstappen

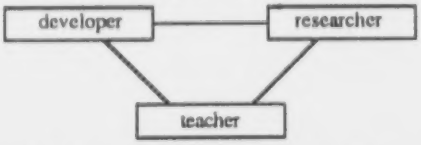
DEVELOPMENTAL RESEARCH ON PAPER AND IN ACTION

The topic indicated by the title is a contribution to the theme concerning the systematic cooperation between theory and practice in mathematics education. Under subject is theorization about the relationship between theory and practice, between didactical research in a broad sense and educational practice. Essentially the theme is a specific case of the more general relationship between everyday and scientific knowledge, as Husserl for instance described in his book on the "crisis of the European sciences" which deals with the problems of the alienation of science from everyday life. I will leave to one side the products of the research, such as learning materials and curricula. Moreover, I will restrict myself mainly to the origin, essence, finality and activities belonging to the indicated relationship.

The activities of didactical research are of several kind such as investigation, development, comparison, evaluation and application. Many countries bring forward important work in this area. Not so The Netherlands, not because we would not want to, - on the contrary I would submit - but we operate in a different manner. The structure differs, particularly the position of Dutch curriculum development, which is unique in several aspects. In the first place a great part of our curriculum development sec has been centered in one institute, which means that other aspects such as research, support, test development, pre- and inservice training have been located elsewhere. This separation undoubtedly has advantages, e.g. a better coordination between subjects is possible, however there are disadvantages as well. It is clear that educational development is thus split up and is dependent on the cooperation with other curricular disciplines. Success stands or falls with the willingness and especially the possibility to cooperate, since time and means must be available in parallel. Secondly, our situation is unique because our work is not primarily aimed at the production of learning mate-

rials, but rather at more global working plans of the schools and at curricula at macro level. The government, who finances us for the greater part, considers the development of learning materials to be a task for publishers.

With respect to the SLO-project "Mathematics 12-16", reasonable cooperation has been established. For some time now we have experienced cooperation with several independent disciplines. It is therefore that our theme - the relationship between theory/research and practice - expresses an additional component. Namely besides a relationship between research and practice, also a relationship between the components of research themselves. To keep matters from becoming too complex, yet dealing with the topic within the given time, I will restrict myself to the triangular relationship



Thus the theme has been narrowed down to the issue of cooperation between developers, researchers and teachers, but that limitation will do for the moment.

Before dealing with the issue more closely, some data about the case. Involved here are six young SLO developers who together form an enthusiastic team, three IPAW researchers from a department of the University of Utrecht and some twelve teachers of whom three work at training institutions and the rest at five practice schools. The research group is financed half by the SLO and the teachers are paid by the SLO in part. Obviously the development group constitutes the heart of the project, although -or rather because- the research group maintains a formally independent position.

The principal goal of the development group is to come to a school working plan i.e. an overall curriculum design for mathematics to 12-16 year olds. The following are their starting points:

- mathematics for all;
- directing mathematics more towards everyday life;

- the presentation of mathematics from contexts which correspond with the intuitive knowledge of the pupil;
- relating mathematics emphatically to the own methodical approach;
- the right to learning, with insight for all;
- a working form which is flexible in principle, but particularly the effectivity of the heterogeneous table group is a subject for testing and investigation;
- differences in progress and levels are accepted while subject matter and pace are similar for all.

The theoretical objectives of the research group (Intra-class Differentiation Mathematics Education 12-16) can be summarized as:
- contributing to the development and improvement of the learning materials
- elucidating the concept "mathematics for all".

The aim is research for the benefit of curriculum development and a mathematical-didactical coloured theory of education, the latter a specific modality of mathematics education for 12-16. Another aim is to trace objective means relationships that are employed in the development of the curriculum in order to derive a descriptive explication of the basic concepts of the curriculum.

Besides the theories already mentioned, all participants are confronted with theoretical views on their specialism. In his thesis, Terwel[1] discerns seven research approaches:

1. Evaluation research (U.S.A.)
2. Naturalistic research (U.S.A;, UK)
3. Action research (Western Europe)
4. Systematic education experiment (Eastern Europe)
5. Design theory/construction research (U.S.A., Belgium, Holland)
6. Education developmental research (Holland)
7. Developmental research (Holland)

In our case it concerns education developmental research which in short can be defined as an informal, incremental, developing approach with strong emphasis on improvement of educational practice and development of learning materials. The researcher

fulfils many roles: developing, training, supporting, investigating. He is allowed to interfere and influence spontaneously and plays an important part in the dissemination of results and the conviction of various groups. The developer's task is markedly creative. He designs new learning materials and tests these for their usefulness. He therefore works in the school. We do not, or rarely try out in laboratory settings. The developer actually gives the lessons or participates in them, in order to be able to observe the actual learning processes from nearby. He analyzes, improves and records his findings in documents which on the one hand contain plans for the arrangement of mathematics lessons and on the other have an inspiring character for teachers, teacher-trainers and textbook authors. Rarely does the teacher explicitly confide his starting points and goals to paper. His inspiration lies most strongly in classroom practice.

On paper, all seems well arranged. Developers and researchers wrote project, planning and execution documents, stated their goals and starting points explicitly and attuned these. Together with the participating teachers and their school staffs, many consultations were held. Most of the documents were cognisable. All seemed to be well, you will notice that I use the past tense. It is not that relationships are poor, on the contrary, one gets to know, respect and understand the other better when working together, yet one becomes conscious of the existence of tensions, not between people, but between ideas, views and conceptions although these cannot yet be made explicit. Participants in education developmental process are not yet really aware of this diversity in views, I believe. Theoretically, one agrees with one another, so it is argued. If in spite of this, something goes amiss in practice, it is blamed on a short in communication. I will elaborate on these aspects "agree" (view), "go wrong" (problems) and "communication", by applying them to the three pairs in the triple relationship: developer - researcher, developer - teacher and researcher - teacher.

1. The relationship developer - researcher

The most important difference originates from the attitude towards theory. Firstly, there are several different views on theory which correspond with the philosophical view on man and society. It seems to me that the most objective form of theory is indicated by K. Popper when he describes theory as that which has been put down in documents, books and such belonging to a third world besides the physical and subjective world. Others share a more naturalistic conception of theory (Quine), "knowledge, mind, meaning and values are part of the same world, and are to be studied in the same empirical spirit that animates natural science." Realism on the other hand, views theory, language and method as the reflection of reality. I prefer the more scholastic view in which theory is considered as a modality, a way of being. In this case, being stands for real, realistic or reality, practice, existing, and such. The modality also encloses the subjective and is a product of the activity. Nature and mode are never objective and depend upon community and the individual. Being is a categorical construct; its context states in greater detail which area of being is at hand. For instance: being human, education research, biosphere.

In this view theory or modality of being translates and explains an area of being with the aid of concepts, judgements and a system of proof. An essential difference now originates, since the translation can have a descriptive or a prescriptive character. In the first case the chosen position is neutral, in the latter it is normative. Theory then must be a standard, a guideline for practice. The difference all boils down to the difference in starting points and goals. With this we touch the essentials which greatly influence the relationship developer-researcher, because already earlier we pointed out that the curriculum is the precipitation of the developer's prescriptions while the description of the researcher has a theoretical elucidation in mind. In a way the researcher allows practice to prevail more than the developer does. Theory arises and emanates from practice, while the developer wants to adapt practice to his theoretical view. If it does not correspond with the imagination or the starting points of the developer he then comes up with

new tasks. The developer does not like to state his aims explicitly behorehand on the level of learning materials. This causes resentment on the part of the researcher who prefers evaluation of practice, i.e. comparison with accepted aims. However, the development of pre- and posttesting does emanate from the demands of the developer. Developers are not readily prepared to drop an once chosen idea, eleborated in learning materials. Naturally they are willing to adapt the material, which often implies extension upon it. Researchers are farther removed from this material and therefore will sooner suggest the elimination of large parts of the package, or the combining of packages. The researcher wants the developer to define the differences in level, to test the work in heterogeneous groups. Is it really true that this didactical approach shows a communication between pupils about the tasks from different levels? The developer already seems satisfied when his material evokes any discussion at all. Whether this takes place at different levels, is something which has a lower priority on his list. An other problem concerns the difference in pace. In a way the researcher's work is more fundamental, therefore requiring more time per package. His results display a time-lag. The developer has already tried out several versions of his own, often in several schools at the same time. He is satisfied. The results of research are just starting to come in when the developer is already working on something new. In other words, the developer's work is broader while the researcher's is deeper. This difference in strategy is a serious handicap for the cooperation because the developer does not experience the positive input of the researcher during his engagement with the material.

In our case, good communication has been established. Both teams meet under my chairmanship, every six weeks. Prof. H. Freudenthal also participates. Furthermore, one member of each team is responsible for communication for the time in between and both prepare the business for the plenary meeting which deals with learning materials, coordination, planning, exchange of experiences etc. Both groups operate in several practice schools of which one is common. It also helps that a member of the research team is originally a mathematician which allows us to speak on equal terms.

Dissemination of the results occurs on a broader front where researchers meet with yet another audience. It is true that this led to a number of competency problems, especially in the beginning, concerning the authorship of modification.
Summarizing the foregoing with respect to:

- *view*
  . common
  . different positions on theory
- *problems*
  . unequal pace
  . being explicit on aims and levels
  . differences in objectives
- *communication*
  . about learning materials
  . about experiences with the children
  . frequently and well established

2. The relationship developer - teacher

This pair also holds that a common view must exist from which to start. The teacher is dissatisfied with the current situation and is looking for something new. He expects new contents and methods to lead to a more harmonious form of education. The theoretical description appeals to him, although much of it escapes him. This means that the teacher has adopted an attitude of waiting in which he wants to see the ideas worked out on lesson level. Originally it was intended that the teacher would cooperate in the development of the ideas. However this cooperation cannot be called successful. The ideas and products which were brought forward by the teachers fit the current curriculum while the aim was to design a new one. In fact, before long teachers restricted themselves to feedback on the presented concept materials. Of course their comments were based on their judgements with respect to feasibility. Most often that meant underestimation. The developer often made several versions of the material before trying it out in the classroom. They

often extended upon these versions, based on their findings, especially on recent experiences in the classroom with tests on the subject. They then modified their actual try-outs. Annex (page 19 to 25) an example of this process. Note the slight input of the teachers. It is true that their attitude changed during the development process. Once they had observed that the children enjoyed these tasks and did reasonably well, they too became enthusiastic. We may say that the teacher team did not function as a development team, nevertheless they were useful in providing feedback and supplying protocols of their lessons.

The teachers regarded the adjustments made by the developer mostly as simplifications or eliminations of the more enjoyable. The ideas of the teachers were rarely used since they did not correspond with those of the developer.

With respect to communication, the following:

- The teachers speak much the same language or jargon as the developers, both use more of an everyday language. All of the developers were former teachers. As a variant on Shapere's "as we learn we learn how to learn": we can say "as we teach we learn how to teach". Strictly speaking there is no gap between the everyday school knowledge of the teacher and of the developer.
- There are numerous contacts, with peaks during the trial periods.
- There is great openness between partners.
- Contact in writing does not work well, the manuals of the topics are read poorly.

Most of the problems stem from peripheral conditions of school education. Especially the underlying hidden curriculum which is manifested during testing, selecting and marking, makes demands of its own. Matters in which the developer is less interested. As I said before, teachers often first of all react by saying that certain learning materials are too difficult for their pupils and then they start working with the material in another way than was intended. The teacher pays attention to the effects on his pupil and on his classes, while the developer is mainly interested in the final version that is ready for distribution on a wide scale. In other words the teacher looks at the local, the developer at the national situation.

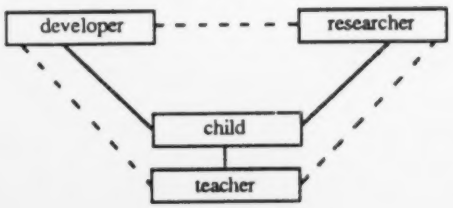
3. The relationship researcher - teacher

We will deal with this relationship only briefly.
View
The views on education are not really different, but barely made explicit and therefore one is not inclined to discuss this topic.
Communication
- On the one hand less simple, because researchers use more of a jargon.
- On the other hand there is more openness about materials. The developer is still more emotionally involved. One is sooner prepared to tell a researcher that a certain topic is wrong.

Problems
- At the outset the participants are not frank with each other. Teachers are somewhat afraid of researchers. The teacher feels that he is being evaluated. Later there is more openness.
- The researcher hopes for the best possible similarity in the conditions at the schools, to make comparison possible. Teachers are not interested in this similarity. So it happened that a certain topic was found to be unsatisfactory. The teachers at the two different schools reacted very differently. There was no protest at one school at all but the work in heterogeneous groups was discontinued to treat the topic in the class as a whole. The other school dropped the topic and looked for other subject matter that was suitable for teaching in groups.

Let me recapitulate.
The activities of didactical research are of various kind. In The Netherlands these activities are accommodated in different institutions. Besides an external relationship between theory/research and practice there is also an internal relationship in research itself. Specifically the relationship between developers and researchers. That relationship is manifested in classroom practice. The theme: systematic cooperation between theory and practice, has been narrowed down to the issue of cooperation between researchers and teachers. It then becomes evident that communication is not attuned to the differences in understanding or lanquage but that there is a

gap with the expectation concerning the usefulness of scientific knowledge. Teachers hide from theory but on the other hand are susceptible to practical learning material. Real communication takes place via the child, i.e. through practice. Theoretical starting points or aims do not penetrate practice, unless evolved from them. It is for that reason that we must extend our triangle.



Teachers, developers and researchers understand each other in the light of practice. In other words, communication takes place via the child in the classroom.

There is another gap between developers and researchers which emanates from the opinions about theory. To developers theory is prescriptive and normative, therefore a guideline for practice, while for the researcher theory possesses a descriptive characteristic and need only reflect modality. Teachers are not concerned about theory, they work and think from everyday experience.

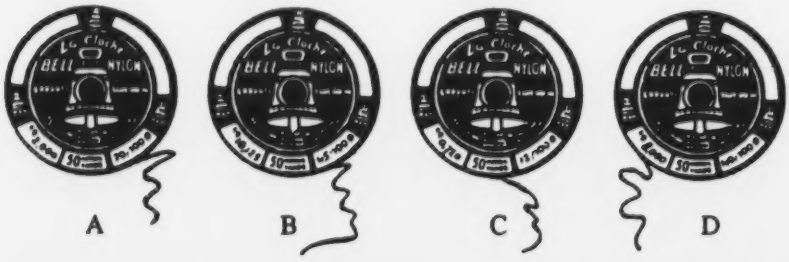
APPENDIX

In previous topics pupils learned something about relationships and graphics in an intuitive manner. The following topic is intended to deal with graphs more exactly. In the first part the situation is described and explained. In the following the data are arranged and for the first time a relationship must be made explicit.

First version (in my lay-out)

Fishing

Fishing line is bought from the roll. You have a choice of several thicknesses of line. Different thicknesses have different tractive power. The thicker, the stronger. Below are some examples of the lables that can be found on the spools of line.

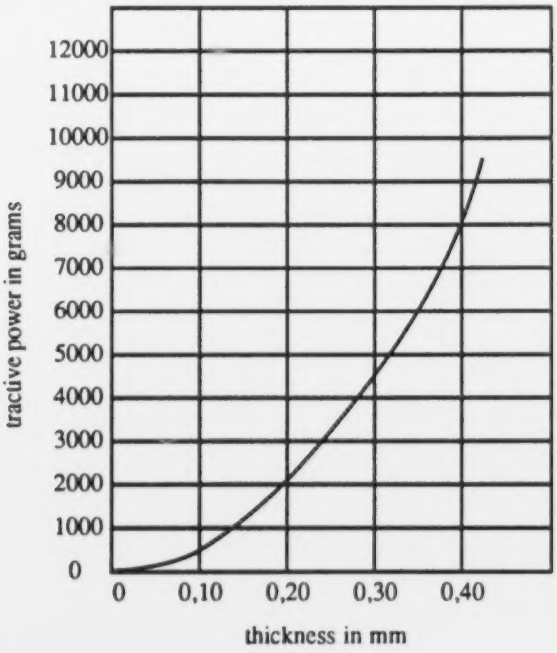


On spool B it says "kg. 1.700". What could this mean? What does "20/100 0 mean?

Make a table in which the following data must be recorded:

line	thickness in mm	tractive power in g
A		
B		
C		
D		

Each thickness has a definite tractive power. Expressed in a graph this looks like



A great deal can be read from this graph.

1. How heavy may a fish be if you are using a 0.45 mm line?
2. A mackerel fisherman mostly catches fish that weigh between ½ and 2 kg. How thick must his line be?
3. Try to find the tractive power of a 0.550 mm line from the graph.
4. Make a statement that is true: if I want a line that is twice as strong it will have to be .....as thick.
5. Do you have any idea of how much you can hang on a lmm line before it will break? How can you find out?

This version functioned reasonably well in the classroom. Yet certain modifications were suggested.
Question 1 was changed because it was not clear. The pupils were inclined to look at the graph for 0.45 mm. The intention was more to aim at the upper limit of an interval at the maximum weight this line could hold. The formulation of question 4 was also changed. Perhaps the interchange of question and answer will give better results.

This is the second version

A great deal can be read from this graph.
Try.
1. If you are using a 0.45 mm line, how much can the heaviest fish that you can catch with this line weigh?
2. Identical
3. Identical
4. Complete a true statement: If I want a line that is twice as thick, it must be .....as strong.
5. Identical.

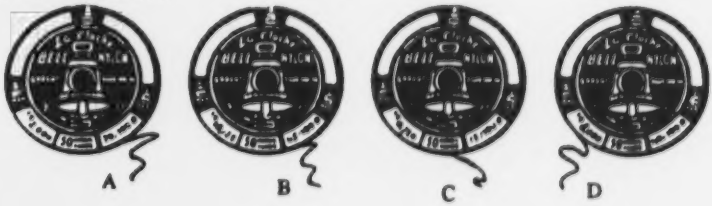
The last question still did not work well. The relationship was not discovered and the table did not function as a useful mathematical tool. In a way this result was disappointing because the developers had faith in the suitability of the context to allow the pupil to invent the quadratic relationship. Next the developers had the idea of introducing a central problem within the context. One that was

intended to arouse curiousity and motivation. They introduced Gemma who was wrestling with the problem: "Can you catch a 40 kg fish on a 1 mm line?" This problem seemed suitable for experiencing mathematical activities. How do you tackle such a problem? Which means can be used? Why do you accept one thing and reject another? In short, the objectives changed with the introduction of new text. The figures were also also adapted; "reality" was not yet quadratic.

At the end of the third version.

"I saw someone catch a fish that weighed at least 40 kilo's, Gemma announced when she came home. "That is impossible, you would need a pole with a thick line just to get it out of the water" her father answered. "Not at all", Gemma said, "I don't know what kind of rod it was but the line was only 1 mm. That is what they said at least."
Her father did not believe it. "According to me that is impossible. The fish must have weighed less. They were exaggerating I'm sure. It is probably one of those strong stories fishermen tend to tell."
Gemma has her doubts now, "Do you suppose it wasn't true". But she saw for herself that the line was very thin. And it did take two of them to pull the big fish out of the water. They had said "40 kilo's on a 1 mm line." Perhaps she had misunderstood them.
What do you think, was Gemma right or not? Gemma is far from satisfied. Let us see what she is going to do. And of course, participate as well. "What different kinds of fishing line are there? How thick are they and how much can they hold", Gemma wonders. She goes to a store that sells all kinds of fishing gear. They sell four kinds of line.

These are the labels.



22

Take a look at the lable on spool B. It gives some information about the kind of line on that spool. "kg 10,125" means that the line will break if you hang somthing on it that weighs more than 10 kg and 125 gram. "45/100 O" means that the line is 45/100 mm thick. The sign O is an abbreviation of the concept diameter...
A few questions:

a. How thick is line C?
b. How many kilo's will it hold before it breaks. How many grams is that?
c. How many yards of line are there on the spool?

Back to Gemma.
I want to know how much a line of 1 mm thick can hold and they don't sell that here. And of course they wouldn't know either. But perhaps this information will help me anyway.
She writes down the information on the lables on a piece of paper.

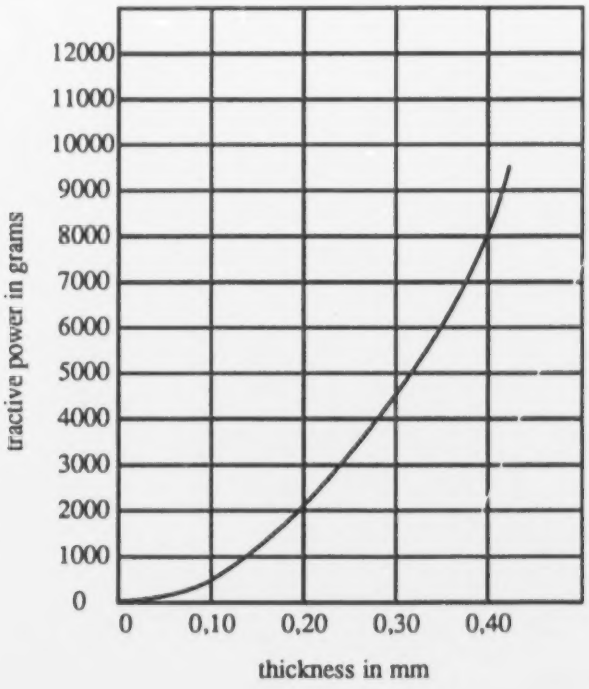
d. Record the information in a table. Write the information for the thinnest line at the top and go on from there.

line	thickness in mm	tractive power in g
C	0,12	720

Using this information I should be able to predict something about other kinds of line, Gemma thinks. It looks like a line of 0,08 mm can hold exactly 16000 grams that is 16 kilo's. A 0,09 mm line will break at a little over 20 kilo's. Well if that is correct I couldn't have been right. Or do you suppose I'm not calculating it correctly?

e. How did Gemma figure this out?
f. Do you think her method is right? Why?

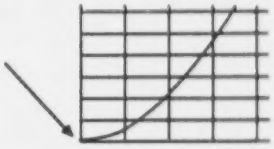
In the library Gemma looked up a book about fishing.
In one of them she saw the following graph:



Oh boy, a graph that shows the strength for each thickness, that is handy. Let's see if it is the same as the line from the store.

g. Indicate the lines from the store in the graph with a dot. Write the letters next to the dots.

h. Why does the graph begin at this point:



You can read quite a lot from this graph that could be of importance for Gemma's problem: whether a 1 mm line can hold 40 kg.

Try to find the following answers from the graph:

1. How thick must the line be for a mackerel fisherman? (between ½ and 2 kg)
2. At how many grams will a line that is 0.50 mm break?
   A few pages back you saw that Gemma made some predictions about the tractive power of lines that were 0.80 and 0.90 thick. You have already given your opinion of her method. Now you also have the graph at your disposal. What can you answer to the following question: were Gemma's predictions too high or too low?
3. Would you dare to predict the breaking point of a line that is 1 mm?
   Back to Gemma.
   "Let's see", Gemma says, "a line of 0.10 mm has a breaking-point of .....grams. A line that is 0.20 mm has one of .... grams. Line that is twice as thick, is .....times as strong. What does that mean? Let's see for 0.15 and 0.3 mm. I think I've got it, because I can figure out how much a 1 mm line will hold like this. After some time Gemma starts beaming. Her conclusion: "I was right after all."
4. Try to find out what Gemma thought of and what she figured out. Fill in the missing numbers and find out whether Gemma is right or not. Write your conclusions and calculations below.

TEN MAJOR ISSUES CONCERNING SYSTEMATIC COOPERATION BETWEEN THEORY AND PRACTICE IN MATHEMATICS EDUCATION

Abstract

In preparation to SCTP-2 the problem field is analysed. The point of departure is the difference of image of man between the theorist and the practitioner. This difference appears from the sceptical attitude of the teacher concerning the benefit of theory. Theory is either a modality of seeing or a modality of practice. Most projects have the functional view, that is, practice can be managed with theory. A gradual formation of theory is suggested by starting with a working model, later turned into a theoretical model. Some forms of cooperation are outlined, which have to be so concrete as possible, in view of the sceptical teacher.

Introduction

In 1984 a mini conference was held in Adelaide, Australia on the "Systematic cooperation between theory and practice in mathematics education" (SCTP). This conference was part of the Fifth International Congress on Mathematics Education and consisted of four sixty-minutes sessions. Bent Christiansen from Denmark was the chief-organizer and also acted as reporter. The report was published in 1985. This paper does not aim to be a summary of that report. For an introduction to this material it is advisable to read the whole publication.

The problem field, -the relationship between didactical theory and teaching practice- was introduced by six speakers from five countries, who deduced their lectures from own experience with research projects in mathematical education. This approach proved to be very worthwhile because of the many good ideas that were presented. However the different situations and objectives of the projects and also the use of personal definitions, barred mutual

understanding. The term researcher for instance, in some views, also stands for teacher or developer. And especially the various meanings of theory were surprising. The phrase which says "a word means what the speaker meant" apparently applied here.

Nevertheless the interest in this topic was relatively great and the request to carry on with the investigation of this topic was generally supported. A sequence of conferences has been planned to this end. The first is to be held in Lochem (The Netherlands) in November 1986. In preparation I do not attempt to present the complete state of affairs, but rather to outline a global structuring of the topic through a number of questions or issues with elucidations. These issues are more or less guidelines for research-reportages or perhaps even for new research. In any case they are suited for a further structuring of the coming discussions on the basis of the many activities which will originate there. Of course this setting of the scene is based on personal interest and understanding of the matter. This paper contains few scientifically founded propositions or theories. They can therefore be adhered to or rejected according to personal experiences and findings. They are given to be tossed around. To avoid one-sidedness, all elements of SCTP are dealt with, though not all equally. Five components emerge

- theory (T)
- practice (P)
- operation (P → T)
- operation (T → P)
- systematization (S)

Each component can be considered from an analytic position, for example from the well-known questions which begin with who, what, when, why and how. In the light of an objective approach the first four interrogative pronouns are less interesting than the last two. It is therefore that each component is approached by the leading interrogatives "why" and "how". Thus ten issues arose and since this is a fine mystical number, no others were sought after.

Let me conclude this introduction with one statement. During the conference last year I tasted an optimistic representation of the

willingness of math teachers. I myself am very pessimistic and I believe our utmost efforts must be directed at the main worldwide problem:

The teacher is extremely sceptical of the benefit of theory and science of mathematics education.

1. What is theory?

The most common answer to this question is: theory is the product of scientific development. This definition immediately leads to many other questions such as what is the contents of that product, described in which terms. A better approach is to indicate the theoretical process, namely obtaining answers to questions concerning reality. The process characteristic originates when new questions are asked about the given answers. That places the question of what theory is, on the same level as: what is reasoning? For a definition of reasoning one needs reasoning itself. It remains reasoning about reason. The same applies to theory. Each theory is based on theory and so forth, resulting in regression. For example, there is a theory X about the nature of the products which can be manufactured according to a certain process P. X as such is an extension of a theory Y for the manufacture of a certain product. Now X is an extension of Y, that in turn is an extension of Z. But what is the limit? Is it an element of reason or of the senses? That means a choice for subjectivism or objectivism. That contradiction corresponds with reasoning versus being or idealism versus empirism and so on. Philosophy has been paralyzed by these extreme views, or to put it less strongly, is marked by the struggles with this contradiction.

The corresponding views on theory are called modality of seeing and modality of being. Nota bene, modality of seeing belongs to subjectivism, because seeing is subjective and hence operates with and depends on the subjective faculties of reasoning and the idealistic view of theory results in hypotheses which can be true or false. The leading principle is determined by the interrogative "why", which is directed towards an explanatory system. The empirical

view of theory resembles a projected map of a certain domain of nature, in our case a map of mathematics education. Here the analogical interrogative is "how". How does it function, move, and so on. Which laws exist? Hence theory is the instrument for organizing experience and for experimental laws and theory aimed at describing the succession and concomitance of events. Language renders the greatest problem. Logic-positivism tried to avoid the language problem by using logic but could not succeed in transforming every theoretical allegation in language merely based on sensual content. Structuralism endeavoured to avoid the correspondence issue by looking only at the relationship and forgetting about the elements. From that viewpoint theory is a compendious but elliptical formulation of dependent relations between observable events and properties.

In each case one must be conscious of the fact that reasoning itself plays a great, even determining role. Already in early times there was a search for the specific nature of reasoning itself. Along which lines does reasoning take place, how does it originate, in short many questions without sufficient answers. Descartes already knew that reasoning systems cannot be reduced to a simple and direct illustration or reflection of reality.

In methodology there is a search for the method of the scientific knowledge, which has rendered a theory of the principles, the means and the procedures of an activity. But as said before, the solution of the question concerning the method depends on the philosophical trend one supports. For the classical German idealism it applies that the dialectic is the general learning method. With diverse viewpoints it is logical that the solution of the problem is sought in the connection between the idealistic and empirical views via dialectic or complementaristic means mediating the "why" and the "how".

The theory must therefore contain

- a number of postulates
- a model or a process of interpretations of the postulates
- a number of correspondence rules connecting the theoretical objects with the experimental notions of concrete phenomena.

Erich Wittmann in his contribution to the mini conference mentioned Popper's conception of "the three worlds". Using this conception he posed that the teacher's practice belongs to world 1, his theoretical repertoire to world 2, while theory in the sense of didactical theories about teaching is part of world 3. Whether or not one accepts this objective world 3, his classification devotes attention to the existence of two theories. Indeed the situation is a complex one. Each theoretician practices his own praxis and inversely each practitioner uses his own theory. So we must interpret two theories and two practices that are inseparable.

2. Why do we need theory?

Aristoteles starts his Metaphysics with the sentence: "All men by nature desire to know", and that is true. Theory is a certain form of knowledge. Therefore the question "why do we need theory?" seems rather ridiculous and subsequently superfluous. But the issue is not that simple. Let me take a closer look.

Theory answers questions about an object. Within the SCTP there are different objects and subsequently different theories under discussion about mathematics, education, mathematics education, didactics, research and the main subject about the cooperation between researchers and teachers. How important or necessary are these separate theories? History clearly illustrates that priorities have been set. Consider the fact that for ages we did not possess any explicit theory of mathematics education. Development of theory did not come to mind for a very long time. Today whole conferences have been devoted to this theme.

It is a well known fact that in olden times theory prevailed over praxis, however theory was taken in the idealistic sense as modality of seeing, as a disinterested attention. The main objective was to augment knowledge and in consequence not to deal with the environment. This situation did not really change during the middle ages because of the unimportance of knowledge in a life that was to be directed towards God. After the middle ages the importance of theory was recognized, evidenced by the saying that "knowledge is

power". Gradually it became clear that *praxis can be managed with science and theory*. Before making theory explicit, praxis unites reasoning and action into one process. Thinkers of various schools investigated the functional character of science in coherence with the pre-scientific experience and thus enforced the idea of managing praxis through theory. Dewey (and other representatives of pragmatism) considers theory and research as an instrument for human policy by which an undefinite situation is made definite. One can find many analogical terms for "instrument" such as constructive interference (Bachelard) or progression (Polanyi) or adaption (Piaget). However the real question is "who uses the instrument". Is it the practitioner or the theoretician or the principal (multinational or the government)? Our original question is therefore not so ridiculous at all in the perspective of the ruling of mathematics education.

The question can also be posed in another manner: for whom does the theoretician or researcher work? Kierkegaard formulated a very cynical answer on this issue[2]:

> The matter is very simple: The New Testament can be understood very easily. But we men are such arrant rascals, we only pretend that we cannot understand it, because we see perfectly well that because it is so easy to understand we would have to act upon it immediately (...) I look it up in The New Testament and read: "If you want to do good, sell everything that you have and give it to the poor and follow me". Dear God, all capitalists, clerks and pensioners no less, the whole race with the exception of beggars would be lost if science did not exist.

In a certain sense the cynicism of Kierkegaard is also expressed by the sceptics with regard to the development of education, where those in favour of certain changes speak about improvements, call it the opposite. This shows once more that terms such as improvement are theoretically loaded, just as the distinction of the very problem itself.

Earlier on it was mentioned that theory as objective knowledge

contains a duality which is expressed in the "how" and the "why". This dialectic of objective and means is characteristic for theory. To put it differently, theory offers both an explanation for a (part of the) discipline (such as for mathematics education for example) and a means for the solving of the problem at hand. Therefore a theory that is found in one place can serve as means somewhere else, whereby the methodology distinction seems to lie in the relationship between objective and means.

3. How to get to the theory (P → T)?

As a whole, didactics of mathematics has a great deal in common with a large number of disciplines such as mathematics, learning psychology, epistemology, sociology and so forth. So in principle there are two possibilities to arrive at the theory: from other disciplines and home-grown from praxis. In the first case the difficulty lies in what to demand and is it useful for, respectively applicable to the theory of mathematics education. In the second case the problems are enormous, stemming from the general interpretation problem.

Let me primarily restrict the elucidation to the second case, therefore: "how to get the theory from practice?" Specifically that means: how to understand children's mathematical reality. Freudenthal formulated a clear opinion on the matter in his didactical phenomenology[3]. First, according to him, comes the phenomenology of the mathematical structures, then the didactical phenomonology concerning the cognitive acquisition within learning processes and finally the genetical phenomenology of the cognitive acquisition in mental growing processes. For the first, Freudenthal argues, one needs only knowledge of mathematics, for the second knowledge of mathematics and of education and for the last one also requires a good deal of knowledge of psychology.

An interesting recent paper by Leslie P. Steffe[4] expresses what inter-disciplinar project work has produced concerning the formation of theory in the field of counting by children. The main problem

is an ancient one: how can researchers and teachers observe the actions of children? Since researchers and teachers use their own (counting) models they cannot well understand what the child is in fact doing. The only way, Steffe says, to understand the child's reality is via the interpretation of the language and the actions of the mathematically active child (directly and indirectly via video registrations). That renders a working model that is continuously and consciously doubted and therefore an experimental model emerges after all, called experimental abstractions. There the crux of the matter is the start from the viewpoint of the child. The experimental model constitutes the essence of the working model. In Athens, Georgia there are experimental models for counting, multiplying and dividing, adding and subtracting. So far the construction of the model takes place within mathematics education. Standard observation, carefully handled, produces the experimental model. The multi-disciplinary project group divides here in theoretically oriented versus empirically oriented parts. Steffe calls the latter, if I have understood correctly, the educational scientists.

Later, in further talks the theoretician tried to understand the educational scientists. For that purpose the video registrations were used time and again. Finally, Von Glasenfeld retired to his couch, shutting his eyes, forgetting about the data, designed a theoretical model. With that in hand the group understood what really happened. This model predicted and specified the activity of a child. It is a theory of counting types.

Using Von Glasenfeld's theoretical model the members of the inter-disciplinary team were able to completely reconceptualize the experimental model of counting and to develop a rather transparant language that emphasized the units-to-be counted. Everyone at least used this model and its terms and the experimental model retained its historical importance.

As far as I can judge, this process is rather unique. The existing counting theory was put aside and the historical genese had to give way to the didactical genese. The new theory was founded on the study of the phenomena in the classroom in the light of learning materials stimulating the development.

Settling a conflict between theories by means of video registrati-

ons! The critical point, however, remains the observations, seeing with your own thinking categories. How can you put aside the old ones? Does holding on to the problem of observing the anomalities not provide the actual counter-examples of the own categories? Here again I come upon the deepest ground, upon the dialectic relationship between theory and praxis.

4. Are there characteristic differences between theory and practice?

This query is important in the light of problems in the cooperation between theory and practice. Besides, it supplies an answer to the feasibility of the teacher as a researcher, one of the possible means for improving the cooperation (see paragraph 8). Also more generally, the lack of characteristic differences opens the door for unifying theory and practice, an important goal of logic-positivism in the first decades of this century.

The main issue concerns the mode of reasoning. Is it true that one and the same mode of reasoning has led the teacher and the researcher to different views on mathematics education or must different modes be distinguished? Of course, corresponding with the answers of yes and no, two standpoints exist based on the relationship between knowing and acting. First, the theoretician made knowledge explicit i.e. separate from the activity, while for the practitioner knowledge is implicit i.e. knowing within action. Secondly, the theoretician's knowledge does not differ essentially from the practitioner's knowledge. Reasoning and acting are always united. The unity of reasoning and handling results within the subject in a gradually enlarging and deepening knowledge that can be translated into language and can be communicated. Now theory and science are more radical forms of questioning and justification than the normal language and action of praxis. Depending on the view on theory (see paragraph 1) one can speak of theory as reorganization, specification, definition and so on. Hence the second standpoint can be formulated as follows: *the theorist wants to know* a great deal more about mathematics education and to comprehend it *with far greater precision* than the man or woman in the class-

room.

It is no good deciding on either of the two standpoints here, but let me look at this matter more closely. A theorist's knowledge deals with a part of the whole of praxis. He is always investigating a small element of the whole. On the other hand, the knowledge of the practitioner is holistic. An unwritten, hidden curriculum exists beside the ordinary curriculum and many apocryphal rules regulate the social situation. Another distinction concerns the language. Theorists communicate mainly on paper, practitioners on the other hand, generally from mouth to mouth. It has already been mentioned that praxis used own theory, interpretations and ideologies. Furthermore practice is enclosed in itself, remains within the school building, while theory is open and works more universally. A very important factor lies in training, but it seems as if both groups are in the same position, namely both the scientific researcher and the teacher possess a subjective methodology and an occupational character. Precisely as for the performance of a trade, it is especially an unconscious skill within a tacit dimension. Thus practice and theory must be learned by doing. No distinction at all with respect to this point so far, but the learning environment is totally different and that leads to essential differences of this training factor. Perhaps there are objections against this presentation because of the way teacher-training institutions function in many countries. However, one is always told that this preparation is not sufficient. Real learning takes place in the trade itself, and is scarcely made explicit and subsequently very subjective.

5. Why is the teacher so sceptical of the theory of mathematics education?

This main issue is not unique. Almost every discipline shows an alienation between theory and praxis, at least in the sense that a deep scepsis exists concerning the useful value of scientific knowledge for solving the workaday problems. The sceptical teacher is in good company, even the famous E. Husserl said: "science is objective, concerning the sense of our existence, science cannot tell

us anything, therefore you have to go to the things themselves. The broken relationship between science and pedagogical praxis points to mediation problems of scientific knowledge to the teacher's everyday theories. Next to the normal causes based on cognitive differences, such as structure and jargon, specific causes can be mentioned based on genese and application.

In the foregoing paragraph 1 it was said that theory is built on theory and that it always contains ideology. Especially from ideology, it is inevitable that several alienation problems result because of its characteristics, such as:

- ideology legitimates the activities of people within a determinate order;
- ideology has to do with interests;
- ideology clouds the real state of affairs or the actual reflection of reality.

Subjective ideologized theory therefore is prescriptive and normative. The products of theorists such as learning materials and textbooks are meant as a guideline for praxis. On the other hand, more objective theory strives for fact and description of mathematics education. In the first case the own ideology is generally not changed. If the learning materials appear to be lacking then new ones are made, also constructed on the same idea. If in the second case the ideology of the teacher does not correspond with the view of the theorist then the described situation will not be accepted as the real one, or will be ignored because it does not touch on the essentials of education. For instance, recent research shows that children, working in heterogenous groups, get poorer results in mathematics. (This investigation was held in the fourth grades of primary schools.) However, the teacher goes on, because he classifies social activity higher than mathematics as such. Here the main, most often unconscious and rarely discussed cause of distinction emerges, namely the *difference in their images of man*. The concepts of the theoretician and the practitioner about children usually differ or diverge. Precisely the image of man determines the success or failure of innovation. All institutes such as education, medicine, or care for the elderly, struggle with the same scepsis towards theory, precisely because of the (supposed) distinct view on man. The

investigator cannot get to know the man or child as intensively as the practitioner. The man-knowledge of the theorist has been constituted more by occasion and is more dependent on time. Therefore the practitioner distrusts man's idealistic abstractive character which becomes evident in the work of the theorist, while he has to deal with nervous little Johnny, the troublemaker Henry and clever, but lazy Mary. Therefore the results of research which are not easily accessible because of the jargon, are not concrete enough for the teacher. Moreover, the theoretical given concerns only a small aspect of education. Usually research goes deep, from the desire to be fundamental but that means a small covering of practice reality. From the teacher's viewpoint his efforts to study and understand are scarcely rewarded.

An important and rather unique aspect of the teacher's hostility - or to put it more mildly - his obstruction towards theory, has to do with the object of the research. In the case of educational research the teacher's behaviour and his activities are essential components of the situation. The teacher is confronted with scientific knowledge in which he participates as an object of scientific reflection and analysis and in which he functions either as a self-responsible subject or as a determined performer of objective relationships. Scientists depart representations deducted from general pedagogical theories, formulated in idealistic terms. Hence as a rule they establish a negative contravention of praxis from those representations. Especially local investigations show a lack of insight into the general occurrence of the contravention. The impression is created that the teacher is responsible for his pedagogical activities. Thus scientific knowledge has become normative and moral judgements have been developed which prevent a discussion about this knowledge which is open and which has results. Another aspect concerns the conditions in the schools. The teacher cannot free himself from conditions such as the rules which apply for changing the curriculum and in particular those which apply to a "secret" change of the curriculum. It is barely possible to change the objectivity of the school and each teacher is involved with it.

An important aspect of the teacher's sceptical attitude was not yet mentioned in paragraph 2. This aspect concerns the definition of the

problem. Usually the theorist or his principal will define the problem that produces two question marks: the formulation-language of the problem - the teacher does not understand the context and the relationship with his work - and the actual existence of the problem. If the teacher does not recognize the problem as an essential and real problem then his interest will soon disappear and he will end up thinking "what's the use, it is only a waste of money".

6. How does the teacher ward off theory?

This matter can be dealt with very easily but is nevertheless important because it is no good doing research with respect to innovation if the teachers can and want to ward it off with great ease. Of course the simplest way of avoiding matters is to ignore it. The practitioner displays no interest at all and avoids each contact. He does not subscribe to a journal, he does not attend meetings, seminars or teacher centres, he will avoid confrontations with researchers and so on. Total ignoring will probably be rare. Usually there is a desire, especially on the part of younger people, to improve their own working situation and to get better results.

If the teacher is conscious of the necessity of changing the educational process he will look at the theoretical possibilities with respect to his need. In one way he is forced to recognize new didactical and pedagogical knowledge. However if he is not confronted with conditions with respect to the use of this knowledge, for example, not a participant in an experiment with obligations, then the new theoretical information will be converted into praxis-theory which suits his own ideas on education. Often the conversion leads to strange phenomena such as the use of learning materials intended for individual and independent work in a classical instructional system. The last example, an improper use of materials, is sometimes the result of a conscious choice and generally the result of misunderstood information. Often not even the manual was read.

7. Why is the dissemination of theory a problem?

This query is inversely intended with respect to the foregoing. Now the same issue is considered from the theorist's point of view. Which methods are useful for dissemination of his ideas and experiences and where do the problems lie? It is superfluous to repeat the remarks which were already made on this point between the lines of the foregoing pages. Let me restrict this explanation - at the same time listing topics for discussion and research - to a brief number of main causes.

The first aspect deals with the proportional number of theoreticians to practitioners. I would guess that this ratio is about 1 : 1000. At any rate, direct communication via personal dialogues is impossible. That is a shame because individual contacts certainly are preferable, particularly in the case of praxis in which communication is generally oral. Direct contact with potential users is the most important approach to directed advocacy efforts.

Besides, the texts are not easy to read, not only because of the jargon but also due to their length. Usually the small kernel of new and interesting information is hidden away in long and boring texts. The practitioner - just like the child - is not used to long texts. Moreover theory is always ideologically loaded. Especially the concepts of the child, although almost implicit, is felt intuitively by the teacher. Political science can teach us to pay attention to saliency, arising from the general situation of consciousness. The ordinary person senses dishonesty. A politician fills a speaking engagement in a catholic area where almost everyone is against abortion. He prefers to avoid this issue but when asked about it he will have to confess that he is in favour of abortion or else his insincerity will be felt. This political rule is often disregarded by theorists. It is often easier to indicate the management character of theory, with the purpose of innovating and modifying education than to tell an untruth about improving education conform the wishes of the teacher.

One of the main causes lies in the understanding of the practitioner; as said before, his knowledge is implicit in his actions. That means that real communication takes place through praxis. Conse-

quently *the practitioner and theoretician will understand each other in the light of praxis.* In other words, communication takes place via the child in the classroom. The teacher pays attention to the effects on his pupils and on his class. For instance it has been experienced that teachers often react by saying that certain learning materials offered by theorists are too difficult for their pupils and that they started working with the material in a different way than was intended. After a while they discovered that their pupils enjoyed the work sheets and easily solved the surmised difficulties. Practice will also illustrate the contradictions in goals such as "the pupil is the central figure" and "the contents is of major importance". Only through practice will the practitioner, experiencing the paradoxical situation, understand the real content of the presentation of the theorist.

In the light of the afore-mentioned communication gap, there are some who are in favour of instructing or developing each teacher to become a researcher. And indeed, if every teacher would be a researcher our cooperation objective would be superfluous. What are the chances? Look at history which provides a great deal of information on an analogical desire, namely the ideal that everyone be an intellectual. In the early days of socialism and communism it was understood that a different definition of the intellectual was needed to abolish the gap between proletarian and intellectual. A new way of reasoning was necessary; not reasoning from above but rather from below, as described by Weitling. Around 1850 this new view - in fact of an absolutely class-free-society - led to the conflict between Weitling and Marx who favoured the philosopher for the good of the proletarian. In Marx's view the utopia was "everyone an intellectual". Marx won and the situation is much the same today. Even now the school system is blamed for its intellectualistic character based on underestimation of manual labour.

What does this lesson teach us? Without changing the idea of research, especially the idea of "each teacher as a researcher" is a utopia. So either change the concept of the researcher or accept both worlds. During the TME-conference in 1985[5], Gerald Kulm said that little is known about the effectiveness of efforts of infor-

mating mathematics teachers about research results and of bringing about changes in practice. If he is right and I believe that he is, then there is a great deal of research ahead of us.

8. Which methods of cooperation are worthwhile?

It has already been asserted that direct contacts are preferable however the unequal ratio also demands indirect methods. Let me enumerate a number of those not disjunct means:

A. Indirect:
   . written material (texts, books, journals, syllabuses),
   . visual material, in particular video registrations,
   . oral material (radio, tapes,...).
   In general this material is provided and distributed by research institutions and publishers, the latter possibly due to teacher associations.

B. Direct:
   . research projects in schools,
   . seminars, lectures and training institutions,
   . reality of the classroom (individual contact).

Of course scepticism is encountered in each of these direct or indirect means (paragraph 5).
This means that a matrix structure can be distinguished with a means dimension and an objectives (or problems) dimension.

Aims

theory				
dissemination				
	classroom	research	seminars	material

means

Here and there results are collected and disseminated. For instance G. Kulm[6] mentioned a few in his references of a review of Beyer and Trice, but the given is scarcely systematized and only covers a small part of the whole range.

During the mini-conference on SCTP in 1984 six papers were presented, all based on experiences with cooperation.

Erich Wittmann dealt with the learning of theory by teacher-students. Starting from two principles: the developing theory of mathematical learning and teaching must be a refinement, an extention and a deepening of practioner's knowledge, and learning is a subjective and social activity, he arrived at "intermediate practice". That means doing mathematics and doing psychology with the aid of teaching units representative of the curriculum.

Mary Kantowski spoke of a teaching experiment in which in-service teachers cooperated with a team of researchers. The topic of investigation was instruction in heuristics (problem solving).

Piet Verstappen reported on the project "Mathematics 12-16" based at the National Institute for Curriculum Development (SLO), in which developers, reseachers and teachers cooperate. Besides cooperation between theory and practice there are internal rela-

tionships in research, particularly the relationship developer-researcher which arises from different goals and products.

Leone Burton described a project which investigated the feasibility of a problem-based approach to mathematics instruction to 9-13 years olds. The aim was also to encourage the teacher as a researcher in the classroom.

Gerd von Harten reported on the project LEDIS at the IDM Bielefeld. This project aimed at developing stochastic learning materials and at obtaining general results on didactical materials. The interaction between researchers and teachers was also emphasized in this project by devoting attention to the daily problems of the school and the classroom.

Guy Brousseau spoke about the cooperation between researchers and teachers in the project at IREM Bordeaux. This project departs from mathematics as interaction and looks at the identification of didactical situations adapted for this view on mathematics and at the communication to the teacher who must work within the limits of a didactical contract, i.e. a set of classroom rules governing the triangle of matter - pupil - teacher.

In comparing this report with the mentioned means it appears that only a part of the matrix has been touched upon. Particularly the indirect means were not handled. Many more examples of success and of failure are necessary.

9. Do we need systematization of cooperation?

This issue has already been introduced in paragraph 2. That which was mentioned there also applies to the cooperation between researchers and teachers.

It is remarkable that in these six reports nobody mentioned the use of knowledge with regard to positive experiences of cooperation elsewhere. It is true that some have heard about negative results but as far as I know a systematized description of adequate researcher-practitioner interaction is lacking completely. In other words the wheel must be invented all over again. It would therefore be very

beneficial for both theory and praxis if the experience would be gathered and made available. Moreover, such a syllabus forms a basis for new investigations to improve this cooperation. Up to this point the whole matter seems rather chaotic.

Is such a systematization possible? Three aspects are important with respect to this question. First of all, is the available information sufficient? Secondly, is the information valid enough to generalize? And thirdly, who is going to finance this work because the profit is more general in character?

It is also possible to give a negative answer to the question of the necessity for systematization. Naturally this depends on the point of view. If one has a unified viewpoint in mind i.e. the dissolving of theory, then in a strict sense you do not really need systematization. From this viewpoint philosophy bears the preponderance of guilt, because instead of digesting the results and transforming them into a new way of looking at the reality of mathematics education, philosophers have drawn even sharper boundaries between science and everyday mathematics education.

10. How to systematize?

To continue from the last sentence of paragraph 9, philosophers have created two worlds so they must also dissolve this mass either by uniting or else by far reaching systematization. However, as remarked in the foregoing, only very little materials has been collected and as long as no complete picture is available, we cannot speak about real systematization. This means that in the first phase the collection and description of information has priority. This information does not have to be structurized. There are certainly theoretical models in other disciplines which are transferable and which can be used after some modification. In any case the adaption means the reformulation of concepts.

Let me stick to the primary form of systematization and formulate some of the wishes.

1. Are there partly useful theories within other areas? Research of successful cooperation can provide a theory which is restricted to the specific situation. Perhaps this theory can be transmitted to other cooperation areas.

2. Do general rules exist?
   Investigation of the available material map prove the existence of -naturally not trivial- laws in the desired interaction, possibly based on well-formulated postulates.

3. Is the whole of the rules and partial theories which have been found coherent and consistent?
   This is not a trivial remark because often it appears that the objectives are contradictory.

At the conclusion the systematizers must indicate their method, i.e. their underlying view or theory as it is hidden behind reorganization, specification or definition of the given theory.

THE DREAM OF COOPERATION BETWEEN THEORY AND PRACTICE

1. Cooperation between two worlds

I want to start the discussion about the cooperation between theory and practice of mathematics education with a few comments on both concepts and their existence. Especially theory is a vague concept as has been explained in 'Ten major issues..'. Practice of mathematics education has more insiders and is consequentely more known. Practice is briefly understood to mean the totality of mathematics education, that is, the whole business as well as the internal relationships and the structures. Off-hand, the game of students, teachers and learning materials taking place daily in school and which has obviously, sometimes clearly defined, often unwritten, traditional rules.

Theory is understood one time objectively, another time subjectively. The one considers theory a way of seeing, as an idea like instrumentalism or realism, the other as something instrumentalistic. The divergence appears from the theoricians: researchers, curriculum developers, philosophers, guides, trainers, and so on. Preliminary I will stick to the vague general of the mixture.

It is important that usually theory and practice are not considered as two phenomenons of one totality. This dates from antiquity and has particularly reference to seeing and thinking. A historical retrospective view brings us on Plato who in fact presents us with two independent worlds in his works the world of ideas and the concrete world.

> "You must suppose, then, that there are these two powers <the good and the sun> of which I have spoken, and that one of them is supreme over everything in the intelligible order or region, the other over everything in the visible region - I won't say in the physical universe or you will think I'm playing with words. At any rate you have before your mind these two orders of things,

the visible and the intelligible."[7]

These two worlds correspond with the distinction between thinking and seeing. In pure thought ideas are exposed while in pure observation the concrete world is realised. In both cases God created everything in His omnipotence and it is up to man to find out. It looks like a game between man and God. The fact that God's omnipotence extends a lot further by having man create his own world, was hardly discerned in former days. As a matter of fact, even today the obligation for man to create is not fully appreciated. Witness the lack of a significant global policy.

The setting of the world of ideas outside the concrete world is clearly expressed in the image of the theoretician as a floating figure.

It is interesting that though Plato is associated with such a sharp dichotomy between theory and practice, his own life exemplifies an interconnectedness that is elusive in his writing. For instance he founded the academy and worked for civil education all his life. Though he is famous as a philosopher and probably the greatest one of all time, what mattered for him was that philosophy rather than being a goal in itself, serves precisely for bettering one's life. Thus Plato repeatedly tried realizing his theoretical, academical conceptions in the most practical settings of all.

His preceptor -Socrates- gave still stronger expression to this attitude through the maxim: "Who knows the good, does it", which also included the reverse: nobody does willingly and knowingly wrong. Knowing is in the might of the good, or perhaps better formulated, knowing conforms to the good and human activity cannot dissociate from it. Philosophizing serves for, as Socrates said, getting righteous and pious with rational sense. For example in Theatetus (172-176) even the world-strange idealist is criticized. Hence it is hard to believe how we have come to see Greek philosophy as only theoretical and dead to the world. Though it is most likely explained by the predominant image of Plato's world of ideas, he himself does not refer to this world in a large part of his work. Nevertheless the idea has long been fettered and stiffened that the platonic static world of the ideas outweighs the dynamic

earth, which is reinforced through many religions. At present it seems that the situation has been reversed, certainly concerning education. Now it is thought that theory moves in the marginality of practice. But appearances are deceptive. Theory fulfils always an important role in practice as is argued farther.

Many of us see today the situation between theory and practice as analogous to the relation between seeing and thinking or between ego and alter ego or between disciplines. These analogies are thought to be merging into new complexes. To see and to think form to know; ego and alter ego the I, and the different subject matters form education.

In fact history shows another process and hence this integrative conception of the two worlds of theory and practice to form a new whole does not serve. There is however, an obverse picture conveyed of the relationship of theory to practice. It is not that these two independent entities join to form a new whole, but rather that an act of disintegration occurs. That is, it is alleged that an already existing world can be split up into several entities. The break-up of a world into theory and praxis would be a wonderful happening. Are there indeed new worlds being created? The falling apart would be the result of a process of idealizations; reality develops into a practical and theoretical component. This view is fortified from considering a domain such as mathematics education in which it is believed that though a theory of mathematics education may evolve, it will be long in coming.

Especially cooperation is the topic of this presentation. Already in general terms there is a lot to say about this topic.

At first cooperation is not something obvious. On page 29 I state that "The teacher is extremely sceptical of the benefit of theory and science of mathematics education". Probably many people will want to accept this statement as a fact. However, the question arises whether the teacher can justify his judgement? Is he likely to be right? Is there legitimacy in his view of theory as an useless product made by floating people. Before deciding the legitimacy issue this sceptical attitude does not promote cooperation, but it does not obstruct it completely, because the relationship theory-practice has

to be seen socially and not as an one-one correspondence (I-you). It is the relationship between two social and structured wholes.
The notion that we are talking of two entities has to be explored further. For at least two substantive entities - in our case theory and practice - are needed for cooperation. Furthermore cooperation assumes a certain independence. If the one is absolutely dependent on the other, then though they work well together, we cannot speak of cooperation in its literal sense, for such an attribution requires a certain degree of freedom at the disposal of each entity. So people can't talk of real cooperation in a situation consisting of two gearing pinions, for one turns round, then the other has to turn also. In short, for cooperation, more or less independent entities are necessary.

I said 'more or less', for though they are not related as two gear pinions, they ought to be mutually involved. Independency has to be coupled with mutual involvement and mutual recognition.
Further cooperation requires an open understanding, i.e. it is characterized as an open rational occurence. Partners have to be prepared to justify their standpoints through verifying validity and truth. Rationality is not to be surjected to emotions and own interest. In an open situation one cannot stick blindly to tradition and one has to be prepared to test tradition.

Cooperation works towards a common goal with the possibility of consensus between the partners achieving it. If both partners relinguish halfheartedly something of their opinions, then consensus is a compromise, which burdens further cooperation. Relinguishing halfheartedly occurs in a strategy of confrontation[8]. The regulative principle of cooperation, inherent to a truly open occurrence, is to prefer. At the end, guided by this regulative principle, there is constituted a synthesis, based on the same mentality. Furthermore the two partners must balance each other. That is not an easy demand because their activities are fundamentally different. For instance consolidating innovation is a task for the people in schools.

The intersubjectivity of theory and practice culminates in dialogue, in which the one knows what the other needs and is able to. He notices the other following, improving and completing him. Writing

is only a superficial, unchanging, unreacting surrogate for dialogue. It is an unresisting prey for ill-disposed people, or maybe worse, it is left unread. The writer does not decide the meaning of his/her text for the reader. Plato himself is sceptical about writing and prefers the spoken word, a medium which liberally allows a change of opinion.

Of course in sceptical situations independency, mutual recognition, open understanding, common goals, consensus based on a regulative principle after a real dialogue are a dream, whereas to cooperate is somewhat paradisic. It recalls a harmony of plants, animals and people in which the one is for the other. Who would not advocate such a thing? Who would not wish a pleasure- garden, in which theory and practice prosper harmoniously? At any rate, we dream of such a cooperation and search for the systematical characteristics, necessary to realise the lived through imagination. A dream has two different meanings. On the one hand we have a dreamhouse, a dream of a child, in general a dream of an object that expresses something delightful. On the other hand it has the meaning of an impossibility, a delusion. Which meaning is at stake here remains to be established in the rest of this presentation.

The issue is to determine how theory and practice relate interobjectively. Till now I outlined especially the intersubjectivity, which overlooks easily theorizing as an always existing process of which theory is always a product. The problem emanates from the apriori fixed characteristics of theory and from the restricted opinion about practice. So I will go deeper into the developmental process.

2. Mathematics as a dialectic process

A good example is delivered by mathematics in which people, only with great difficulty, relinguished the basic idea of mathematics as an independent Platonic world. Many mathematicians really believe in a domain of real numbers which shows its qualities in perfect contemplation. The fact that this idea is naïve, becomes clear in the investigation into foundations. The proofs of many so-called funda-

mental qualities, like the thesis of middle value, appear to be logically deficient.

A century ago this point of view was never expressed, for people accepted intuitively principles such as continuity, functions, averages and so a proof was not needed. Today we know that the definition of a real number is the product of man made qualities, that are considered desirable and not the other way round. So it is just the thesis of middle value which also determines the real numbers. The real numbers do not exist a priori, they evolve by way of a process, which unrolls itself within often unconscious rules. What has been stated here about the real numbers, applies to every mathematical object.

To every object a context of activities has been coupled, that is to say a whole of human activities which can be carried out with this object. With that a paradox is attained, for the object itself also changes in the activities. The essence of a bicycle is that it is movable. As soon as this object is fastened however it changes its nature and even name: a home exercise machine. A hammer is used for beating on something. The modern hammer of a carpenter has been manufactured in such a way that nails can be pulled out with it, so at the same time it can function as a pair of tongs. Perhaps hammer-tongs would be a more appropriate name.

In spite of these concrete examples the actual happening still remains rather hidden. The negative numbers made somewhat clear that mathematics is neither an act of unifying two already worlds nor one of dissassociating a prior world. It was also revealed that the changing context of activities forms the essence of a concept but in what way does such a thing happen?

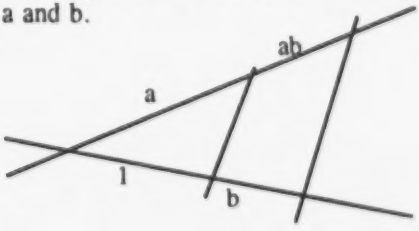
The historical development of mathematics and natural science lends itself to find that out, and that development is characterized by progress that is more haphazard than we realize. At one point growth almost stands still, at another there a big leap forwards. One can postulate that evolution is based on revolutions, in which certain difficulties are settled. A couple of examples illustrate that point. Parmenides claimed that the All, this is everything which exists, is one. Zeno on the other hand, a disciple of Parmenides,

came upon the impossibility of the existence of the multitude. A distance, so he argued, cannot consist of an infinite number of parts, because then Achilles cannot catch up with the tortoise. Dividing as a real activity has a limit, following the argument. Here the partition between seeing and thinking comes home to roost. How does Plato understand Zeno's problem? Here he claims that Zeno's analysis depends too much upon sensory perception. In relinguising sensory data, that is in relinguishing the real world, Plato calls for the *dialectic* as a way of resolving conundrums.

> "So when one tries to get at what each thing is in itself by the exercise of dialectic, relying on reason without any aid from the senses, and refuses to give up until one has grasped by pure thought what the good is in itself, one is at the summit of the intellectual realm, as the man who looked at the sun was of the visual realm"[9].

Go back to the ideas, to the models according to which the things have been formed. *Coming off* the senses is the first phase of dialectics.

The Greek failed to achieve such a total vision with numbers. They could not abstract the dimensional component. The multiplication of numbers as a matter of fact increased the dimension.
$x^3 + x^2 + x$ made no sense in antiquity, since volume, area and length cannot be added. This situation remained unchanged for a long time.
It was not until the 17th century that Descartes made an important step forward in working with relations, although the geometrical context was not yet abandonned. Using a unit length as given, he was able to find a length (not an area) that was the product of two given lengths, a and b.



In using proportion the product ab retained the dimension of a and b. In a way the ab has been concretized at the same time. The diagram expresses the proportion and ab.

The negative numbers formed an even more obvious barrier. In relation to debt one could operate with it for quite a while, but not without a good deal of resistance. Even Pascal posed that, if one has two and has to give up four, one is left with nothing. For ages the biggest problem faced was the multiplication of negative numbers. The real world had to be abandonned in order to be replaced by the relational thinking. In 1809 Ludlam still fought against the use of negative numbers, because, as he argued, smaller: bigger can never be equal to bigger: smaller, like in -1 : 1 = 1 : -1. In short the geometrical interpretation emerges time and again.

The view of the negative number as an equivalence class of pairs of numbers meant a gigantic step forward, which only a few could make. What has been presented here in main features for the infinite and the negative number, applies equally to other barriers like 0 : 0 (Leibniz), extract the root of a negative number (Gauss), the existence of several parallels through a point (Lobatsjewky), the dependency of measures (Einstein) etcetera. The breaking through such barriers - particularly empirical barriers - opened up at the same time magnificent and powerful areas of science like the differential calculus, the complex analysis, the non-Euclidic mathematics and the relativity theory.

Though these examples correctly support the hypothesis that evolution comes about through revolution, that point of view tends to hide a more substantial one. That there are continously emerging barriers in itself is not surprising. Science is bound to encounter problems and it takes time before such problems have been resolved. In different words, progress comes by fits and starts. No, the essential point is not to be found in the consecutive problems, the attention should focus on the nature of the barriers. Time and again the empiric has to be overcome. But how, an alert individual will observe, can one time and again abandon the empiric? One should do so! Here our scholastic way of thinking emerges, which is dominated by the impossibility of contradiction. Of two contradictions either the one or the other is true. This is the basis for the

indirect proof, which states that if the one is impossible the other must be true. Such a scholastic thinking accepts the world as an inflexible given. She is present as an unchangeable object of perception and reflection. Only the dialectic as highest form of human activity comprehends the essence of that object. Indeed, in this perspective the abandonned empiric does not return and only theory is to be developed.

3. The three phases of cooperation

It was pointed out that dialectic starts with coming off the senses and after that theory has a clear field for development. Hence the dialectic process has two phases: coming off and developing. But there is something very remarkable: namely the construction of new barriers. Take the non-euclidic geometry, which axiomatic goes against the euclid geometry. Cayley proved in 1859 that all objects of Euclid geometry are to be expressed through objects of projective geometry and that in this way the projective geometry contains the Euclid geometry. The metric, a modification of distance, can be introduced in different ways. In 1871 F. Klein demonstrated a projective model of non-Euclid geometry, which is the inner of a circle. The chords, without endpoints, are the lines. The length of AB = $|\ln(ABCD)|$, if AB is chord CD. The duplicate proportion (ABCD) defines the metric and with that the new geometry proved to exist.[10] That means the dialectic process has to restart.

In the example of the evolution of mathematics it became evident that the object is wrapped in dimensional chains, which define the context of activities that can be performed with the object. In this way $x^3 + x^2$ made no sense to a Greek. His praxis of mathematics did not allow this addition. The theoretician, concludes that practice only develops herself inwardly and what is more important, he discovers a boundary which praxis will not step across. A boundary which has its natural ground in the experience of discipline as a unity, in close coherence with the context of activities. Objects belong together, form a unity once they show the same proportions

or rather once the same activities can be performed with them. Note here that two limitations have to be watched: a synchrone one and a diachrone one. The first is based in essence on the goal-means conversion - what has been explained can be used as a means for the explanation of an analogous problem - the second is based on the historical limitation - the means may not be used outside a defined area (for instance negative numbers may not be multiplied). One could imagine praxis to be like a circular area with holes which can be filled synchronally, but that may not diachronically be stepped out of, since at that moment one enters the unreal. Praxis is marked by a confined reality, operating with concrete activities and concrete objects within historical bounds. Increasingly theory is aware of its proper task in recovering its origin and liberating itself from the antique unworldly. With the Greek theoretical knowledge was made of extraterrestial substance, a kind of epistomological ether, like celestial bodies were supposed to be made of particular imperishable substances.

In the middle ages all knowledge was derived from devine wisdom. Human ratio had little value, the learning and the praxis had to surpass itself unto the Most High. Descartes stood at the entrance of a new area in attaching high value to reason and science and to accept mathematics as a scientific method. Intuition as the light of reason brings certainty to man, according to Descartes. This certainty is looked upon quite differently nowadays. The scholastic world was a fixed one, true enough moving, but always emerging on the begin, so things move in circles. The rise of natural science and its empirical thinking did not essentially change this fixed world. Only the outcomes of thinking were no longer certain, because experiments are necessarily limited.

Karl Marx investigated especially the changeability of living conditions. Starting from economical and historical research into the so-called original accumulation of capital, he observed a mechanism in the economic development of the world. Friedrich Engels summarizes Marx's finding:

> like the small business by its own development, that is
> through the expropriation of the small owners, created

> necessarily the conditions for its own destruction, the capitalistic mode of production likewise has created the material conditions which will lead to its annihilation[11].

This historical process lead to a rather diverging view on dialectic, which Marx referred to as: negation of the negation. The insight of the development running within a narrow natural confinement is the heart of this dialectical process. Small business resting on private ownership allows only a limited production. The negation of the negation restores in the economical praxis the individual ownership on the basis of what is achieved in the capitalistic period.

It is interesting to apply Marx's dialectic to the praxis of education. In the past education was an individual affair, where father and mother taught the norms and values of society and provided an occupation. This individual education had its natural confinement and gradually school as an institution evolved replacing for the greater part parental education. And likewise the school as institution will be destroyed by its own confinement and individual education will return but at a higher level. Indeed, one can observe everywhere the decay of the school and evidence is growing all the time, that micro-electronics will create the opportunity of individual education. But here we are speculating towards a total negation of the praxis after a long term. It is better to return to the real nature of the praxis in continuous change and the task and position of the theory herein.

The Hegelian dialectical technique of thinking, by continuing the epicurial, democritical line of thought took an impressive step forward by accepting the world in overstepping change. From there praxis is in transgressing genesis, and theory and dialectic receive a new meaning. Theory and praxis are no longer fixed and definite, but rather situational. Theory opens up the diachronical border of praxis by exposing the historical determination and extends the empiric with new relations.

Specific objects are unchained and taken in their context. The latter is a necessity since the empirical support at the same time formed its chains. Furthermore the contextual properties constitute the new objects. Take fractions, negative numbers, complex num-

bers or possibly less trivial the vectorspace (Q, Q). Then what surprisingly happens is, that the originally pure theory has been taken over by the praxis. Praxis absorbs theory and is subjected to a metamorphosis. The old praxis has gone, a new one reveals itself. The old algebra went under into the complex algebra, operating with new algebraïc numbers, which loosing their relational status, became equally real. Since for the practitioner algebraïc numbers are equally real as real numbers.

The given examples, differential calculus, complex analysis, relativity theory, are also no theories in the idealized meaning as they possess a real meaning in the eyes of each scientist. This being so, praxis has moved the diachronical borderline and the game can repeat itself, in short a continuous development. Lektorski alleged the same recently, specifying theory into epistemology:

> The epistemological reflexion appears to be an irreplaceable means to break out the limits scientific mental clichés and to penetrate into new dimensions of objective reality[12] [13].

This remarkable happening rests on our economical, constructive capacity to use the existing situation for the future and indeed as economical as possible and integrating the old one. Precisely as we put together multiplicities to an unity and determines the dialectical at once of the unity and the multiplicity, like we can put under a symbol *n* all numbers together with an element of those[14].

The dichotomy between subject and object, between the knowing person and the known is a typical Western fashion and knowledge based on this dichotomy too. But knowledge works with building elements that belong to knowledge itself without being a tautology, because the metaknowledge is taken into knowledge. The epistemologer has to be conscious of the suppositions, to which belong the merging of metaknowledge. For instance Theodor Wiesengrund Adorno disapproves the separation in method and object. Human knowledge and epistemology are one in Adorno's view, so epistemology itself is a part of human knowledge. Adorno objects the dichotomizing of reality into two independent, autonomous parts and a separating element as being from nothing. He thinks that the separating in pairs such as subject and object; nature and history;

form and content participates in reality. Separating does not happen from a thinking, as the classic philosophy supposes, outside reality. You can not prove through thinking, that thinking is the origine of everything else, says Adorno, with which he reproaches for instance Husserl's absoluteness of thinking.

In summary, it was shown that Plato was always looking for the fixed divine reality, for the good in itself. His philosophy was founded on mathematics and reversely, this philosophy influenced deeply mathematics by shifting hypotheses. Furthermore he elaborated Heraclit's idea that the elements must enter relations, that is, the foundation of relativism.

During the following ages the idea of relativistic thinking developed, but gradually and with reactions. Now we have three phases of dialectic and particularly of cooperation:

- coming off praxis
- developing theory
- realizing theory

It is necessary that theory is regularly unlinked from praxis - including the praxis of theory - and consequently points to new ways, which at first glance look to be unreal. Strictly speaking theory and praxis never cooperate, it is the end of a dream. Praxis can absorb theory but in doing so changes itself.

praxis → theory → praxis at higher level

In fact I posit the following proposition:

structures become experience.

Consequently relativism is part of reality, or put it differently:

reality is in the making.

Hence realistic mathematics education is only empiristic mathematics education. In education is making an appeal to reality, an elitism unless this reality exists for everyone.

In the development three phases are to distinguished. First, the experiment is less important. The real cosmos is static, which structure is to find in logos, so by dialectic thinking. Second, at the end knowledge is based on experience. and mathematics and logic are means to describe. Third, knowledge is a construct of human mind, which says only something about the relations of reality, it is

the liberation of absoluteness. Corresponding, philosophy has to be directed:

1. The static world, to produce knowledge of the Good.
2. The dynamic world, to produce verified knowledge, i.e. logy via perception.
3. The dynamics itselves, to produce functional knowledge, i.e. gony via relativism or maybe relationism.

Concerning education as praxis the consequences are important too.

4. The ideal image: teacher-researcher

The scepsis of the teacher to theory is in essence evidence of a lack of insight in the fact that his teaching is charged with theory. Through the ages, education has been nourished with ideas about its fabric. Only research into the history of our education explains the present situation.

Such explanation is of immediate need in understanding the diachronic. It is common knowledge that the work of a scientist in for example the 16th century cannot really be understood without knowing on which this scientist based his knowledge, without knowing on which trail of thinking he was travelling. Likewise the present teacher cannot be understood without the historical context.
Some view the teacher as researcher at the same time. Actually it would be the ideal situation. This thought is likely to have come from the idea of the failing implementation of the theory. The theoretician usually lacks a good deal of facility to move forward and once he is succesful he has to reach out to the teacher-practitioner to convince him of the profit in stepping out of his routine. The teacher-theoretican is naturally not troubled by this problem though he is bound to a conservative circumstance, e.g. the so called hidden school curriculum. One can not expect, more or less from every practitioner to function as practitioner-theoretician.

The actual problem of the teacher-theoretician is more profound and rests first on the distinction between synchrone and diachrone development. In the synchronic education is changing albeit not essentially since it remains within a developed context of activities.

Examples: simplifying existing procedures, use of different models, a different sequence of certain elements of subject matter, removing or adding certain subject matter. It involves improvements which almost naturally appear in the next edition of the educational book - not only of the textbook - which title or structure needs not to be changed. Such changes require research of course, which can examine the effectiveness of the changes. The teacher improves as a matter of fact his own praxis, but remains in essence within its realm. In fact (s)he does not essentially come off praxis. Does one distinguish here between the better and the ordinary teacher? This "better" teacher often receives no or little academic backing and has usually to be satisfied with support from his colleagues.

However, there are exceptions, who actually come off their practice, who are inspired by something, who show a different teacher/theoretician, operating clearly diachronically fundamental upon ideas of Clairaut, Piaget, Freudenthal and others. Here it involves a true crossing of borderlines. Here different images of mathematics appear and - what is more important - different images of man. Precisely the latter requires a great deal of intelligence and flexibility, otherwise one cannot surpass and reflect on present man, otherwise one cannot achieve new insights corresponding with a new human. These exceptions really achieve the second phase and develop a piece of theory. Only in a dream one can expect such vision and ability from all teachers.

Also the third phase, the realizing, is usually too difficult for the common teacher. The greatest obstacle is formed by the examinations, on which society, governement and others have a strong hold. An individual teacher cannot change much here.

Changing educational activities radically leads to other humans or with less deference to other products. It is exactly this on which theory has to be judged, because theory rejects praxis and wants to bring it into a higher level. At this point the dialogue in itself is insufficient. The practitioner in principle does not understand the theoretician. In my contribution to the first conference it appeared that the theoretician and the practitioner can only communicate in actual practice. That is through application, through classroom and

pupil activities only the teacher really understands the value of what is new. An essentially different concept of fractions, the substance of mathematics for all, working in heterogenous groups, all this is only understood in the new praxis[15].
Let me return to the beginning to the paradisic situation. In fact the cooperation between theory and practice goes very well in picture, for paradise is essentially not so sweet; the main activities are to eat and to be eaten.

At the end this presentation is on the one side rather global. Mathematics education was taken as a complex. No distictions were made such as a division of activities in a learning and a didactic component or a division according to schooltypes. Of course the process of learning depends of such parts of the variety called mathematics education. On the other hand the demands on theory and praxis are very clear, such as theory has to function in a truly border-crossing fashion, and hence has a prescriptive character. In this scope theory as more of the same is a waste.

The consideration of theory as a means leads to theory mainly conformed to praxis and so a theoretician has to care for mathematics education. The border-crossing task includes or demands real attention to the direction of the change, for this direction is not aesthetical objectively defined as has been thought throughout all ages, for instance by the Good (Plato), the Perfect (Erasmus), the Beauty (Whitehead) a.s.o. Mathematics education has in my view an emancipating function, it serves a better world for all. This task is still heavily underestimated, although a increasing interest in the relation with the image of man or worldview is to be observed. Too little attention is given to the big dilemma that the intellectuals decide what intellectual cultivation and particularly mathematics educations contains.

ON LEVELS OF INTERACTION BETWEEN THEORY AND PRACTICE

In advance

Up to now there have been three conferences: Adelaide 1984, Lochem 1986 and London 1988, all devoted to the relation between theory and practice. The formulation: 'Systematic Cooperation between Theory and Practice of Mathematics Education'(SCTP), led already to misunderstandings during the Lochem conference. At first sight it was based on the use of the words 'cooperation' and 'of', but the misunderstandings were really fundamental. Especially the native speakers argued that 'cooperation' was only possible between people. For categories like 'theory' and 'practice' the term 'interaction' should be used. During the discussion there appeared an unwillingness to use the concepts 'theory' and 'practice' as such, which was not the case for the subjects 'theorist' and 'practitioner'. So the main difference concerned the recognition of objectivity of theory and of practice. The phrase 'Some people do not like theory' expressed the background opposition empiricism - idealism, sometimes represented as the English school versus the German school. Of course not every Englishman is an empiricist. Some thought that theory arises from practice, which is the comprehensive process from which theoretical moments emerge and derive their function, and as a process constitutes the unity of theory and practice. Others thought that theory is a science in itself or a philosophy or an ideology or that it is identical to didactics. Every term of SCTP can be interpreted in a different way and unfortunately this kept on happening during our sessions.
A few individuals used the word 'coordination' instead of 'cooperation', which increased the confusion. This paper tries to analyze this matter and recommends a solution.

1. The functional conception of theory

Theory has a bilateral character. First it can be autotelic, that is, a modality of seeing or a perspective, leaving practice unchanged or liable to autogenic changes. The other side is the functional one. It is the idea that practice can be improved by theory, even be managed with theory.

In both cases the theory-practice relation is at least temporarily global, precisely as the relation between everyday and scientific knowledge has become more global, conform the change of substantial thinking in relativistic thinking. We realize that theoretical concepts are not names of things, rather relationships between objects, just as we realize that one's problems are integrated into world problems. Man no longer sees to all of his needs himself, his food, culture, etcetera. Knowledge is no longer related to doing in a naïve manner. This implies that the individual must move beyond his local boundaries and must use the pared-down knowledge from above at least to the extent that he can operate on an equal level with others.

Throughout history the structure of theory has received particular attention. Nearly always the structure was seen as a kind of mathematics, or even as mathematics itself, for instance Descartes thought so. For Plato the structure was logos and for Aristotle logic. Spinoza came back with geometry. Husserl desired a non-theoretical description of phenomena, a back to the original experience. What is more important, of course, is the world view that led to such points of view as cognition via anamnesis or, not to go so far, cognition with the help of a priori concepts, or the empirical view that cognition may only make use of perception and experience and must repudiate all a priori elements thus permitting the possibility of controlling the world only through reason.

The justification of actions then is not in argument but in reason, or to go even further, in practice, in actions themselves. In short, the whole scale of philosophy, in particular epistemology comes into play. And scale implies conflict. Truth is not to be found in the visible world, according to Plato, a point bitterly contested by the

empiricists. A more subtle issues is the distinction between knowledge and capability and the cognition in capability which is closely related to cognition and capability in activities for all. Essentially it's a question of the position of the logos.

Many educators are content if their students can act, for instance can count. The fact that they understand little about it, such as the dimensional character or the clustering of units, in other words, have no cognition of counting, matters little. Let me illustrate the various currents by using the notion of "number". In order to count this concept is essential. To knowledge belongs that the order in which this counting occurs is of no consequence. This knowledge is comprised in the concept of "number". Young children do not know this concept, which is evident in how they learn to count.

Cards are spread on a table. "Count them", they are asked, whereupon two five year-olds begin with difficulty. The one answers 12 and the other 15, and both are pleased with their answers because they arrived at them differently.

How does the meta-knowledge of counting-way-irrespectiveness come about? The constructivist, as a good Kantian, will ascribe this pure concept to reason which is present a priori and can be used. The Platonist will argue that this concept can be called up by means of anamnesis. The positivist has a problem; he can only make the law of the conservation plausible by inductive means. The pragmatist will simply practice counting until no mistakes are made anymore and will not explain this "number" concept. Thus, there are many ways to introduce or to teach the meta-knowledge that the sequence is not important and thus irrespective the sequence, the result is always the same. As long as this is not the case, children will respond to the question "How many" by counting. "How old are you?" is interpreted as "How many years do you count?" and the answer is logical: 1, 2, 3, 4, 5.

Consider counting as practice. It is only naming and theory in practice is telling about the way the things had been named. Pay attention to the word 'in'. This theory is purely describing the doing in a naive manner. As soon as theory wants to explain it has to hypothesize. In this case the associative law is the main funda-

mental rule. After founding counting on this law it is fundamentally changed. Theory changed counting and has a functional kind. Then theory, coming off practice, is a theory of practice.

Concerning algebra as a complex of mathematical operations theory can be in algebra itself, but also of algebra, i.e. its structure or its applications. Hamilton expressed it as follows:

> The Study of Algebra may be pursued in three very different schools, the **Practical**, the **Philological**, or the **Theoretical**, accordingly as Algebra itself is accounted an Instrument, or a Language, or a Contemplation; accordingly as ease of operation, or symmetry of expression, or clearness of thought, (the **agere**, the **fari**, or the **sapere**.) is eminently prized and sought after.[16]

The agere is directly related to 'activity' and the sapere to 'object'. This implies that in the first case the relational active character is stressed and in the second case the substance. Both are to be unified in mathematics (algebra) as a cognitive system of pairs (object, activity), that is analogous to language. In language subject and object are related through the verb. A sentence, besides place and time aspects, has the form: subject - activity - object, abbreviated S-A-O.

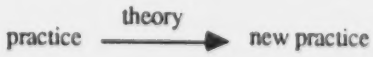
Further one can also consider S-A-O globally, in which O is for instance a social or a physical dynamic system, such as mathematical education. Then an activity A is a purposeful, resolute change. Usually the system is stable for a period through locking in a perturbation. Think of a marble in a bowl or of a white blood cell absorbing bacteria. Nevertheless there are shifts occuring in jumps from one structural balance into another, such as genetic activity and learning activity. Between the entities are a lot of interactions, which form the very basis of activity. So generally A consists of a composition of real actions, or operations or mental actions. Also A consists of actions that run completely automatically or according to strict fixed patterns.

In a living system at least a part of the actions does not run automatically. Interactions are sequences of actions induced by subjects. Generally the interaction is the relationship between the h_{1i} induced by S_1 and the h_{2i} induced by S_2, thus $h_{11} \rightarrow h_{21} \rightarrow h_{12} \rightarrow h_{22} \rightarrow h_{13} \rightarrow h_{23} \rightarrow \ldots$

However the h_{2i} does not always react on the h_{1i} and it is obvious that in such a case one should not talk about interaction but about activity. So $A = \Sigma_i h_i$ and particularly $A_1 = \Sigma_i h_{1i}$ and $A_2 = \Sigma_i h_{2i}$ are two different states of A. In fact, in the case of activity we cannot pinpoint causal chains determining exactly the changes in the structural balances.

The key concepts 'interaction' and 'activity' are often confused. Sometimes 'interaction' and other times 'activity' is taken as fundamental concept. For instance, in historical materialistic thinking interaction is the basic concept. One starts from the interaction between man and environment of which purposefulness is demanded. If the aim is the changing of this environment then the purposeful interaction is called activity.

In my opening address at the Lochem conference[17] I dealt more deeply with the relationship between theory and practice of mathematics education. Since an absolute theory and an absolute practice never exist, I outlined the difficulties in talking about cooperation. I called the cooperation between theory and practice a dream.[18] Only a temporary theory cooperating with practice as such does exist. For the rest practice is imbued with theory. In a way the situation can be compared to the predator and the prey. Most of us don't see this as cooperation. Cooperation shows three phases: coming off, developing and absorbing into. Strictly speaking theory and practice never cooperate; practice absorbs theory and in doing so changes itself. In a diagram



Both theory and practice are always in the making, which shows a complementaristic entanglement of structure and process. During the development of theory they remain separated. The development of practice happens synchronously and autonomously with the means of practice itself, but does encounter into barriers. It is precisely theory, raised to a metalevel, that pinpoints the barriers, that investigates their characteristics and deduces the possibility of

overcoming them. Sooner or later practice will extend through absorbing the overstepping and will function imperialistically through simply absorbing theory. Theory becomes practice, provided it is a good theory. I gave a counterexample: the 'middenschool' (The comprehensive school) in the Netherlands remaining at the theoretical stage. The incorporation demands at least two conditions:

1. It has to be a good theory; if the analysis is inferior and if the real barriers have not been indicated or real solutions have not been found, then eventually a communication breakdown exists and the theory cannot be successful.
2. Theory moves, I argued, on a higher level and has its own structure and jargon, which naturally leads to a chasm between theory and practice. In my Adelaide paper[19] this situation concerning mathematics education is described and explained; that is theory is only to be understood by the teacher in the practice of the classroom.

In short every good theory starts from practice and ends in another practice.[20]

But practice must be open to criticism or to consider new ideas, else the situation is stable, which does not mean immobility. In the unstable equilibrium a small power suffices to change radically the original situation. For a system as mathematics education this unstability does not occur really, however for the individual teacher the situation is quite different. In case of real cooperation practice changes almost always intermittently and rarely continuously.

Hamilton's tripartition corresponds to the simplified streams in the opinion of men and the world:

-idealism/structuralism;
-empiricism/pragmatism/instrumentalism;
-complementarism/dialectic.

Karl Jaspers mentioned characteristic features of what he called the scholastical, the experimental and the dialectical thinking technique.[21] The problem is that each of the indicated terms is connected with an historical context. It is of importance whether the dialectic in the platonic, the hegelian, or the marxist meaning is taken.

In marxism the human attitude to reality is essential, that is the meaning of the practical, objective activity, aiming at preconceived changes. Activity is overemphasized and has to constitute the unity of subject and object. *Theory is the ideal plan of this activity as well as supposition and condition.* Theoretical activity is entirely at the disposal of praxis. In other words theory is plan, condition and supposition of praxis. Summarizing this marxist view: Theory is continuously revealing, theory opens up new horizons for practice and is always of use to practice.[22]

Of course the positivist and the structuralist consider theory differently. So according to the position of theory and practice there are two opposite views on mathematics:

1. Mathematics as activity. One could say mathematics as practice and theory is plan,
2. Mathematics as the result of some activities. Now mathematics is theory and practice is application.

Mathematics as activity is pragmatic by nature, conversely mathematics as a whole is idealistic. It is to be said that the dialectical stream lies somewhere in between. The situation is similar to language, however less obvious.

2. Theory of mathematics education, some aspects

Until now the terms 'theory' and 'practice' are used as categories without defining them. Since they are idealizations I shall not define them. So practice is a fundamental concept. I might say: practice is the application, practition, execution of instruction or theoretical knowledge or rules. However as soon as ones tries to go on with defining applying, practising, and executing one reverts to the practice concept itself. Detailed specializations are more profitable such as the practice of life, a teacher, a lawyer, a doctor etc.. In general I will use the practice of an activity or of a complex of activities.

In education the following trichotomy is interesting:

the practice of learning;

the practice of teaching;
the practice of researching or developing.

In our case practice is met on three categories each of the form S-A-O.

1. A pupil is working at a task/text
2. A teacher is instructing something
3. A researcher is analyzing what is happening in the classroom.

From a dialectic view S-A-O is considered as a trinity i.e. an element is considered within the relationship. Hence look at the entangled relationships:

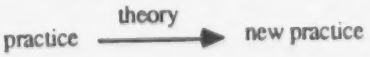
r_1: the pupil copes with material
r_2: the teacher brings about r_1
r_3: the researcher improves r_2

The product of functions $f_3f_2f_1$ furnishes a model of the entanglement.

This entanglement appears in the use of not always synonymical terms, such as cooperation, relationship, collaboration, activity, interaction and coordination. In each category the right term has to be chosen. As long as it concerns relations of the kind r_1 and r_2 we speak about the relationship T-P in mathematics education, otherwise it concerns the T-P of mathematics education.

During the Lochem conference this matter was discussed and it was decided to change the of to in, hence the title of the Report is now: SCTP in mathematics education. At that time I thought it inopportune to make an issue of it, however I did not heartily accept the change. Fortunately we did not really exclude the researching - teaching relationship!

Now I arrive at the relationship of theory and practice. As long as theory has not been integrated into praxis the question of relationship between theory and practice is significant. Look again at the diagram



In new practice the theory is implicit and there is not a theorist. Then the knowledge is in activity itself. Theory not integrated

contains knowledge of activity. That is theory is on a **higher level** above practice. The theorist knows something of practice which is not in practice itself. Conversely one can only speak of a theorist if (s)he raised (her)himself above real interaction, that is into mental activity. It is in simple words the shift from being able into knowing. Nevertheless it is a shift into another type of knowledge[23], because theory yields information on practice. In essence the dialectical relationship between knowing and being able is continuous. **The key words 'knowing' and 'being able' represent theory and practice (T and P) for each activity or complex of activities.** To illustrate this here are some examples:

- r_1: - algebra as Hamilton indicates;
  - $P_{1:}$ counting, $T_{1:}$ establishing the impossibility of continuing naming, solution the introduction of a place value system;
  - P_1: multiplying as algorithm, T_1: when and where multiplying?
  - P_1: The Euler-Poincaré characteristic for a manifold. T_1: for which general objects is this characteristic applicable?
- r_2: - P_2: instructing as procedure (voice, length of sentences, duration, etc..) T_2: when do you instruct and which ways of instructing exist and how are they related to learning situations?
- r_3: - P_3: being able to observe a classroom situation or being able to produce learning packages or explaining a learning theory, T_3: indicating the basic principles or working out the historical development of the concept 'continuity'.

A mathematical algorithm belonging to a fixed, special object lies behind a theoretical investigation of the general objects belonging to this algorithm. Conversely a mathematical object belongs to or defines the congruence transformations. The first is algebraic, the second geometric, and in algebraic geometry the dialectic is to be found.

A metaphorical research of general objects going together with a fixed A in S-A-O is typical for the theoretical activity as well as

determining the restrictions of A-O. *The crux is that two levels are involved and that is the main problem of the theory - praxis relationship.* Praxis is empirical and theory is on the metalevel.

Sometimes this situation is very complex such as: Given a nice package of learning material tested on a number of students, for which student is this package suitable? Of course the number of students in our world is limited, nonetheless the determination is practically impossible, so a lot of theory is needed to answer this query. The main difference with mathematical examples lies in the impossibility of introducing imaginary students.

What matters is the difference between knowledge **in** an activity and knowledge **of** an activity. In essence it is the hoary difference between insight and knowing what to do. The first is more formal and fundamental and belongs to the mature, emancipated, responsible human, the second is more intuitive and belongs to bondage.

Generally the terms in S-A-O are unequally valued, which leads to dominations. The rest of this chapter is briefly dealt with the successive domination of the object, the subject and the collective subject.

In traditional mathematics education, education is seen as the transfer of a knowledge-base. It is a view which is somewhat conceitedly described as the 'broadcast metaphor'.[24]

Repudiating this traditional concept would be much more easily accomplished if the relationship between theory and practice was better known. In practice, it has proven to be very difficult to effect a different education model. The cause is self-evident. In traditional mathematics education knowledge is practically identical to its form. Knowledge does not need to be structured since it is already structured. Teaching may therefore be limited to explaining.

The counterpart of this structure concept is process orientated tuition. Education at this level is a system of interactions. In other words, the communication between all participants is an integral part and the contents have been put into the background. Education is a series of processes, in which the subjects negotiate on interpre-

tations, intuitions, purpose and so on. In this process the contents must be developed using questions and hypotheses. The conduct of this education process is no longer solely controlled by the teacher. Of course the teacher poses the problems but how and if they are worked out is up to the pupils. This freedom, though partial and limited, goes against the historical confines of education as an objectified form of knowledge and is therefore a substantial concession. In process orientated education the contents are subsidiary to the aim of training and the pedagogical objective. Already people in general formative education take the direct meaning less frequently as a guideline, so the transition is not so prodigious.

Distinguishing between knowing and being able has developed two opposing forms of education each with clear shortcomings and is closely related to the image of man. Often there is more concern with the ability of the lower classes as opposed to their knowledge. In the process orientated concept of education the pupils do not have at their disposal the means to develop the contents. The liveliness of the happenings cannot disguise the fact that the pupils often feel utterly helpless. They have no grasp and miss the general point and the overview. In the other case, where the understanding of the presented knowledge is the issue, the knowledge loses its guiding, developing function of initiating the being able. The teacher conveys the information and the pupil only has to digest it.

The solution lies, according to historical and dialectical materialism, in activity as the explaining principle, which is a general characteristic of human world and awards humanity its position in the real and ideal world. Only in activity subject and object are no longer opposing but interacting. With this interaction, the one is continuously the assumption of the other. It is a circular interaction. So it is not about the way the absolutes, such as structure and process, context and activity complement each other but about elementary complementarity.[23]

In general, knowing is developed from reflecting on the conditions of being able. The interaction is the epistemological principle that constitutes the object. Therefore new objects are created upon which the existing being able can be applied and so knowledge is

expanded.[26]

The original marxist view tried to understand education and pedagogy socially. From this, about 50 years ago, new forms of sociology and economy have arisen. Central to this is the domination of the society, the collective subject, so that teachers and pupils are merely represented as objects of the social process. Subjects are the forte of humanity, in other words the qualities or general characteristics of humanity as a species, such as capability and autopoiesis or self control. Both characteristics can only be realised in relation to the community. In this view activity is developed with the terms from and with the help of a scientifically viewed world derived from materialistic sociology. Representatives of this trend often couple the concept of activity, either through deduction or through analogy, to the political-economic concept of labour. This reduction to the concept of labour leads to impoverishment.[27]

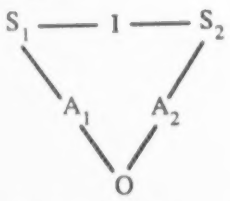
Activity theory, based on historical materialism, excludes empirical and epistemological theoretical dimensions. What remains is a one-sided 'social-scientification' of the being/consciousness contradiction and of the natural sciences and humanities. This one-sidedness is dangerous according to Sandkühler and many others. In their eyes, man as the subject of the activity or labour, has a far more independent position and one should separate psychological and anthropological statements. In short, the problem can be centred on the subject of activity. People who describe subjects with socially limited powers of action and therefore go from the assumption of the normative function of solidarity in the activity, are in opposition with respect to radical constructivism, the theory of self-reference depending on individuation. With an abstract epistemological subject concept the activity theory would place itself along the same lines as other processes which lack a subject, for instance biological evolution and structuralism. Both absolutes lead to a crisis of the subject.

3. Levels of interaction

Education has in mind the development of pupils and it depends of its philosophy whether solidarity or individuation are the results aimed at. I propose reserving the term 'activity' for the relation between a subject and an object S-A-O and the term 'interaction' for the relation between two or more subjects S_1-I-S_2 and eventually for the relation between the subject and the little fellow -homunculus- in the subject S-I-S_r, as a common language for the inner speech. Then interactions I_1, I_2 and I_3 exist in each category r_1, r_2 and r_3. Take I_1, the interaction among students is the standard interpersonal example. But what about the activities between: the knowledge and ability of a student, or a student and his text? The first is intrapersonal, and the second is impersonal.

Interaction means in a literal sense 'mutual or reciprocal action', that is, both operators influence each other and change in consequence. But in the example: the student is learning the text, the text is unchanging. Only the context of the text can change, so the coherence, the connection in which the text has been situated can change during the studying. The student asks his neighbour something, or his teacher, or he refers to a dictionary etc. The text as such remains unchanged.

In a scheme:

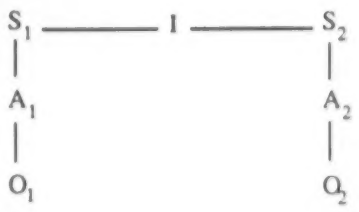


Also I consists of a composition of real or mental actions or operations. Generally the interactions are the interpersonal or intrapersonal relations.

In this case the activity is the learning of a text. This activity connects the student with the text and so with an object. One might reason that the learning text has been produced by one or more authors and that the activity is interpersonal namely between the student and the author(s), but it does not really change the situation,

for the text itself remains unaffected.

When students have different tasks or work at a different pace on the same task the following scheme arises:



The intensity of interaction depends on many facets that call for the continuous attention of researchers. For instance the following should be separated:

1. S_1 and S_2 work individually and at the end the one justifies his/her activities - he/she argues or proves - to the other.
2. S_1 and S_2 work individually, but there are moments where they compare the results.
3. S_1 and S_2 react incidentally to each other.
4. S_1 and S_2 coordinate their activities continuously and alternately.

It's not easy to characterize true interaction on the same task, however the following points should be present:

1. The actions have to be directed to the same aim. This aim is reached when there is a direct consensus on it. Whether or not the real aim has be achieved others must decide.
2. It is a joint action, both receiving their turn.
3. There is talk of mutual recognition, without domination. Everyone must be prepared to defend their points of view intellectually, without over bluffing but using proper arguments.
4. There must be openness for what the other has to say.
5. The actions must be perceivable. Especially, talk and action must go together.
6. There is mutual understanding, where the one understands what the other does and says, in any case saying or showing when he does not understand.
7. There is freedom of action, in other words no stimulus - res-

ponse situation, also no apriori holding on to fixed procedures, so no deterministic linear thinking, but at least a structural thinking consisting of free, creating, creative activities and an exchanging of the latter.

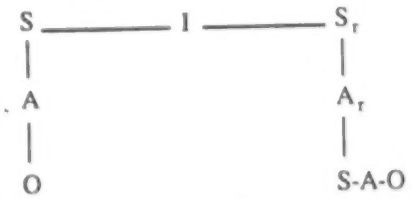
8. There must be a recognisable working method - a plan - having reference to three dimensions: procedures (algorithms, beginning with data analyzing in relation to the question, restating of the task, suggestions towards solutions etc..), mediating images (symbols, patterns, models and general objects) and a theory (synthesis of rules and principles).
9. The taking into account of each other demands tuning of the tempo, so the one waits for the other, after his analysis of the task, in order to discuss each others findings etc.. It is characteristic that every now and then there are pauses.
10. There have to be emotional expressions of pleasure and astonishment. Aversion is allowed although not always. This is a difficult factor, because the pleasure and emotions of the teacher differ from those of the machine and the pupil. Emotion is inherent in the way aesthetics is viewed, looking at beautiful things. This emotion only appears when people sense the elegance of mathematics.

The intensity of interaction depends also upon what has been allowed or obliged. Today students are sometimes allowed to communicate and less frequently are allowed to work together. However dialogue is a necessary condition for investigating and confirming one's own thoughts. Vygotsky developed his theme concerning the internalization of dialogue into inner speech and thought and generalized his findings to a rule of developing psychical functions.[28] He alleged that interaction between man and fellow man c.q. environment is necessary for learning and leads then to man's development.

Just as important is the complementaristic opinion that can be heard in Vygotsky's work in which he rejects behaviorism through denying a merely direct relationship between stimulus and response. The stimulus- respons process, Vygotsky says, runs via signs used as tools, I would say via structure. In this way structure and process are inseparably connected.

The real crux lies in the possibility of the coincidence of S_1 and S_2. In other words the subject has an internal mutual action through reflecting on his/her actions and from there performing the next action. So S reflects before going on. That is, the scheme S-A-O is too simple.

More appropriate is:



The intensity of this reflective interaction is the qualifying factor for development. A scarcely reflecting student stays with his knowing in activity itself and will not arrive at the level of knowledge of activity.

Not everyone accepts the intrapersonal knowing, the coincidence of S_1 and S_2, through which activity and interaction have been separated. In that opinion interaction is not a subject-subject relation on activity, but interaction is a part of this activity. In this view knowing is always interpersonal and the other is essential for activity. For instance Stieg Mellin-Olsen alleged recently:

> Activity theory is ... about the formation of the social individual as a part of his society.[29]

The title of his book rests upon this, for:

> In its broad sense Activity is a political concept.(p.37)

As soon as we admit the term interaction, we consider alternately S as subject and object. That implies the accepting of the objectivity of the epistemological subject, i.e. we accept the self-developing 'I' continuously distancing itself and always knowing that it knows. Thus we take a complementaristic standpoint, recognizing the indissolubility of the structure-process relationship and the inevitability of the control function of the context. The latter applies particularly to instruments and tools of the activity, for the meaning of an instrument depends upon the context. Now by means of the key concepts and the principle of complementarity I am able to introduce levels of interactions.

Previously I argued that the knowing - being able relationships are more important than the objects as such. I dealt with r_1. r_2, etc.. and within the r_1 there are the r_{1i}, ... Each interactive relationship is a complementaristic entirety of structure and process, maybe better at knowing and being able. The means of knowing are introduced on the basis of shortcomings in the being able or as happens more often on the basis of the economic principle, like the place value system or an indirect proof. The concepts of knowledge are indirectly created through criticism of the being able and conversely the knowing directly influences the being able itself. Now if knowing sheds insufficient light on being able, then mainly process actions remain. In this situation I would like to call interaction **cooperation**. On the other hand, if a high knowledge level exists, then structure co-ordination is possible and I would like to use the term **coordination**.

Summarizing, interaction has been reserved to intra- or interpersonal activity and shows two levels: the lowest cooperation and the highest coordination.

Often activity and interaction appear confused. Here is an example:

> Teacher: "How many times does two go into twelve, Joachim?" The boy thinks and thinks, moving his fingers, but does not answer.
>
> The teacher, observing the confusion: "Okay, here are some blocks, can you count them in pairs?"
>
> The boy with fore and middle finger repeatedly pushing away two blocks counts: "2, 4, 6, 8, 10, 12"
>
> Teacher: "Now again, without blocks, how many times does two go into twelve?"
>
> Joachim: "Two", simultaneously uplifting the fore and middle finger, "four", two more fingers are raised, pause, "is two", looking at his fingers and remarking the double use, "six is ... three" and after a while, "twelve is seven".
>
> The same question for An and Clara. After some fiddling around, An: "If you count up to twelve, then I will count the number". In close harmony they find the answer six,

and more importantly they both understand the procedure more easily.

Examples:

Category 1: The cooperative, compliant pupil imitates exactly what is demonstrated and does what has to be done; the coordinative pupil knows with insight, looks to himself for examples and applications.

Category 2: The guiding teacher according to traditional methods follows strict procedures; the coaching, coordinating teacher allows pupils much freedom and allows them to invent and discover.

Category 3: A curriculum institute is more cooperative, while a university institute is more coordinative and pays more attention, for instance, to the historical context. So they have different views on theory. A good example is the difference in functioning between the Institute for Didactics of Mathematics of the University of Bielefeld and our Institute for Curriculum Development[30].

4. Development of research/developmental projects

In the foregoing I looked mainly at the teaching - learning relationship. Now I will go into the researching - teaching one. Practice exploits means in order to change and theory can be an effective one. In this perspective theory as such is senseless. Theory has to be a means, of course always in connection with practice. It is the perspective of the teacher/practitioner. Nevertheless, usually the teacher is sceptical of the benefit of theory and science of mathematics education, sometimes because (s)he does not want to change, sometimes because (s)he does not understand the message.

The teacher's sceptical attitude towards theory is thus based on his

or her own implicit philosophy. For this reason Leone Burton rejects 'cooperation models' such as 'researcher as a teacher' and 'researcher uses teacher'.[31]

The view of theory as such is essentially structuralistic, theory has been detached from the theorist in the bureaucratic way of thinking that dominates our culture nowadays, at least in the western countries.

Also where theory has been absorbed by practice, the theorist remains, hence a practice of theorists always exists, an ideologically loaded practice. So the question of the ideological changes in research and development projects is meaningful with a special philosophical point of view that these projects may change teaching practice. Let me illustrate this with a simple example. A theorist T believes in mathematics as a means for adult life. In this view mathematics education has to be realistic. A teacher P asked his class to solve the following problem: Someone is 100.000 hours old, is she older or younger than you are? Later the theorist T said to P: 'that was a silly assignment, it is not realistic, nobody will express his age in this way! 'Maybe you are right', P answered, 'but the pupils enjoy solving this problem'. P did not change his lesson, but what about T? I think T will modify his view by incorporating a stronger rule for selecting learning material: the degree of attractiveness of the assignment. During our sessions learning from practice was a common thought.[32]

Changes in teaching practice take place in several ways, varying from heteronomic to autonomic, from what has to be done to self-modification. For the latter a mirror has to be held up. A number of cases presented at our conferences were based on this autonomical change, such as interviews in which episodes are used or lesson transcripts or videotapes.

Showing alternatives takes the middle position between heteronomic and autonomic changing, for ignoring really positive alternatives does not square with responsibility. But how to evaluate the alternatives? It's rarely so that they are completely transparent and that their philosophical grounds are well known, certainly not if the

information has been given via methods, theoretical rules or slogans.

Strangely enough during the conferences we did not speak often and profoundly about our own ideological changes, although a lot of cases were presented. Mostly we heard about the project ideology itself, but not about the development of the ideology. But there were exceptions, and then the atmosphere was more emotional. It happened once in Lochem after a frank introduction that people conflicted with each other. A situation complicated by the different philosophical views on mathematics, mathematics education and man. It is a pity that the reporter of this session was somewhat phlegmatic and laconic:

> The question was raised, that the participants have very different interpretations of what is meant by "theory"; a lot of scientists would not call the theoretical orientations of projècts theories (it is too early to formulate theories, eclecticism is the answer to this state)[33].

I recall this as a proof of the differences in state of the cases presented. In 'Ten major issues' I said:

> Theory is built on theory and always contains ideology. Especially from ideology, it is inevitable that several alienation problems result because of its characteristics, such as:
>
> > ideology legitimates the activities of people within a determinate order;
> > ideology has to do with interests;
> > ideology clouds the real state of affairs or the actual reflection of reality.
>
> Subjective ideologized theory therefore is prescriptive and normative. The products of theorists such as learning materials and textbooks are meant as a guideline for praxis. On the other hand, more objective theory strives for fact and description of mathematics education. In the first case the own ideology is generally not changed. If the learning materials appear to be lacking then new ones are made, also constructed on the same idea. If in the

> second case the ideology of the teacher does not correspond with the view of the theorist then the described situation will not be accepted as the real one, or will be ignored because it does not touch on the essentials of education.

I think most of the projects are based on subjective ideologized theory, which is strong enough to resist local critics. Besides they are usually short term projects. In any case too short to change the main ideas. Often an independent evaluation is lacking.

In the enumeration of important influences on educational developmental projects five stages emerged, oriented on:

a. Piaget, Bruner, the English Nuffield project and so on,
b. Gal'perins learning theory,
c. Davydov and Wiskobas,
d. Problem solving,
e. Constructive activity.

A lot of participants recognized their own development in this list.[34] The transgression from one stage to another is caused on the one hand by dissatisfaction about the results in practice and on the other hand by the presentation or offering of 'major theories', which give knowledge a new intellectual power.

A difficult and confusing point is that one dominating ideology does not exist. Hence mathematics, mathematical thinking and mathematical didactics always have to be distinguished. Properly the situation is much more complicated. Maybe a study of the influence of, for example, the ideas of Gal'perin or Piaget on research and education could deliver more insight and could be of great value in the light of the epistemological borderline between ideology and science.[35]

The three phases of cooperation, inherent with the functional and explanational character of theory, end also to *the model of three phases of the development of projects*. What applies to practice analogically applies to theoretical practice.

At first its philosophy is rather naive.[36] The theoretical knowledge is then implicitly present in acting, it's a knowing in activities.

In this phase the researcher or developer is not really an expert. One has to make a distinction between 'being an expert' and 'working as an expert' as Michael Otte pointed out. Indeed the teacher is an expert, e.g. of hidden dimensions in the classroom, likewise the pupils who often know very much about classroom problems, know more than their teachers. The naive phase is characterized by an attitude with which the members of the research or development projects are completely caught up in practice. They express this by calling themselves 'participating' researchers or developers.

A second phase pertains to the question of what everything is exactly. It is **practice-ology** as it were. Here the members are no longer participating only in practice but also with detachment. In the second phase theory and ideology have to be worked out, and the researcher must be able to operate in such a way that he can discover the truly prevailing opinion of the teacher. It is the phase in which a metastandpoint is taken, which can be described as **knowing of practice or knowing of activities**. A situation quite similar to what Richard Skemp calls 'knowing how' as opposed to 'knowing that' of the naive state. Now studying practice produces expert knowledge, e.g. information on teaching styles. In this phase the difficulty arises of what theory really is and how it is connected with ideology, with the whole of ideas or world views and philosophy concerning life and mathematics education.

The third phase, the **functional phase**, shows a manner of acting and thinking, of which neither the detachment from practice, nor the directly being caught up by it is central. In this phase the relation between practice and theory is central rather than participation and detachment. In functional thinking the relation is characteristic of both the researcher/developer and the teacher. In this case scepticism of the teacher is out of place. Now the teacher keeps in touch with theory to enlarge his or her knowledge of mathematics education, without any chance of being manipulated.[37]

To avoid misunderstandings I would like to stress that also in the first and second phase functional aspects can be discovered, however the relation theory-practice has not been structured then. Hence the model is not a psychological but a structural one, in which the

operationalism of practice and theory has been arranged. Nevertheless a similarity with the open and closed attitudes of mind is unmistakable.

In conclusion, what about the ideological changes? Where are we going to? I outlined a phase three, depending on a more objective attitude. In this phase both the theorist and the practitioner are looking for a suitable ideology. Therefore they need a structured relationship and an awareness of all possible ideologies and their implications for education, students and society. They also have to be aware of how ideologies are hidden in education. Then they will no longer be victims of their subconscious ideas, but will have surpassed them. That is the way we have to go. However it is a long way and usually our projects are not given the necessary time.

Summary

The relationship between theory and practice as a fundamental concept has been investigated by looking at activity as dialectic. Each activity or complex of activities has a practical side. In relativism theory bears new perspectives for, and becomes enveloped in, practice. It is useful to reserve the term **activity** for the relationship between subject and object leaving the term **interaction** for the relationship between subjects.

Interaction in education occurs on two fronts especially, between a pupil and his contemporaries or teachers and between a teacher/practitioner and the researcher/developer/theoretician. The differentiation in levels of interaction in mathematics education can be based upon the difference between **knowing** and being able, where knowing can be seen as a raising above being able. The levels relating to this are coordination and cooperation.

The change in the practice of mathematics education ranges from heteronomic to autonomic; from what needs to be done to self-changing. Holding a mirror to your own work has proven to be suitable way to effecting self-change.

The interaction between researcher and teacher often consists of

three phases or, in this case, levels. Firstly, the researcher participates in classroom activities and keeps his theoretical views and ideologies implicit. Secondly the researcher exceeds the classroom activities and expresses his theory and ideology. The not directly involved teachers are on both levels usually quite sceptical. Finally, in the functional stage, the relationship is central and the interaction is for everyone institutionalised. The researcher complies with the wishes of the teacher leaving no more room for scepticism.

THEORY AND PRACTICE IN MATHEMATICS FOR THE LESS ABLE

Abstract

The development of theory shows a growth coupled with a purification of practical elements. This dual process led to the opposition theoretical thinking - positivistic thinking or existential/postulational thinking - constructivistic thinking. Positivistic thinking departs from experience and is based on common sense. This division is closely related to images of man, to earthly bonds on the one hand and on the other to mental freedom. Growth and purification produce the theory-practice problem in two essentially different formulations: what is the use of theory and how is theory to be applied?

Between theory and practice are the relations, which are the instruments of theoretical activity. The crux is the ambiguity of the instruments, as they are to be considered either positivistically or theoretically. The positivistic interpretation gets entangled in the particular. Besides a didactical approach grafted on empiricism or realism is less economical.

Less able students show disturbances of sensibilities especially qua behaviour and theoretical cognition. The SLO project OWI has been based on the idea of hyposensibilization. This idea has been elaborated in a number of principles. To avoid the ambiguities of models, the principle of elementary complementarism has been introduced, together with a revision of mathematics as an activity in the light of relational mathematics. The complementarism has been illustrated with an example. Unfortunately the ambivalence of instruments gives the practician the opportunity to use the instruments in a manner not squaring with the intention of the theorist.

Introduction

Some time ago I developed a learning package about measuring for the less able students. It was the fourth package for the first class. The previous packages dealt with the pocket calculator, the counting of large quantities and playing games. In this paper I tell something about the theory and practice of the first task of 'Goede Maatjes' a title which cannot easily be translated, as it means both 'Good Mates' and 'Good Measures'.

To understand the development work I will first present, in a very concise way, some characteristics of the school and students.

Formulating characteristics for the feeble student means generalization. The individual student will not possess all the characteristics. Besides a global characterization is relative with respect to country, region, school, age, sex and so on. What in one country is considered as less able may be found in another country as normal or even good.

In our country there are schools and school departments for individual vocational training for 12+ years, of those who are low attainers in a number of subjects or who are different qua social and emotional behaviour. At present we are developing a new curriculum for these students. The translation of the characteristics of the less able into an adequate curriculum is in this specific case definitely a difficult task. These students:

- differ strongly qua age;
- come for the greater part from environments with little schooling;
- are twice as likely to have come from non-native origins;
- a third do not come directly from primary school, but from special schools;
- have not much perspective on the labour market;
- differ vastly in cognition and behaviour;
- are rather ego-oriented;
- are superficial, casual, unconcentrated, dependent;
- are linguistically deprived, have a small vocabulary, formulate and communicate badly;
- have a fear of failure coupled with little ability for self-correction

and self-control.

The present education is generally rather mechanistic and especially directed to arithmetics according to the programme of the primary school. In fact the less able have to learn again what they did not learn before.

For them arithmetics is end-education. Other parts of mathematics do not often appear.

On average the teachers are not badly motivated, didactically, and take much care of their students. Most teachers are not so mathematically trained. Many are unsatisfied about their lessons and are open for new perspectives.

Bright students may even progress with bad materials and teachers, but less able will not, which makes the development project so attractive. Inferior materials, e.g. based on wrong ideas, will always fall short.

1. Hyposensibilization

From the first global analysis of the situation I concluded that we have to follow a totally new avenue. The teachers and the authors of the textbooks are experts in their domain. One may assume that they, from their principles, make the best education. Now that they, in spite of all that, experience the situation as a deadlocked, indicates the necessity of a radical change of course. Their principles are mostly implicit. An outsider, which I certainly was, will notice them earlier. I thought the main guide principle of their didactical concept was taking the students under protection. One fits in with the characteristics of students by problem-avoiding education, so:

- individual, isolated education, students are not allowed to talk and work together;
- linguistically deprived education;
- strong discipline;
- learning through imitating, and so on.

The situation resembles the common therapy of allergy. Avoid all allergic sources. The best thing is to stay in a clean, sterile room

and live like a greenhouse plant. Against this therapy there is hyposensibilization, that is, gradual habituation, so that one can stand the allergic matters.

In the same way, our OWI-project offers such an educational alternative on the basis of **hyposensibilization**. The sensitive aspects coherent to the characteristics, are justly not avoided and presented in increasing dosage. The idea is that those students are gradually able to cooperate, to do something independently, to become more linguistic and can handle the freedom. But they have to learn it and so need to be offered the opportunity.

Furthermore it is thought that realistic mathematics education does not work, as these less able students have a basis of orientation which is too small and are less prepared to learn what is real and to think mathematically instead of positivistically. Besides they cannot choose often the right one from the variety of meanings, which diminishes continuously their self-confidence.

What does hyposensibilization mean with regards to the given characteristics? In the temporary curriculum 'OWI in de kijker'[38] I go into all aspects, although not always deeply. Here I must partly confine myself to a synopsis, in which students characteristics are connected with education. Only into a few aspects will I go more deeply, sometimes even deeper than in 'OWI in de kijker'.

1.1. Arithmetics as end-education

Remedy **Mathematics** consisting of
- Arithmetics (20 %)
- Measuring (10 %)
- Algebra, functions included (30 %)
- Geometry, both planar and spacial (15 %)
- Statistics, combinations and probabilities (25 %)

1.2. Fear (of failure), unconcentrated, casual

Remedy **Playing games**

Two or more people playing. Playing occurs usually intensively.

1.3. Less adapted to school-learning, slow formation of concepts

Remedy **Natural learning**

Serial learning is unnatural.

At present the teaching concerns with a lot of small consecutive problems.

Typical is the mania for tests. A small part is thoroughly taught and the result is immediately tested. On the other hand they are becoming acquainted with concentric learning without the demand of thoroughness. The object of the activity will often return and is developed in the course of time.

A great difficulty arises from the increasing complexity of the world, so that we cannot manage problems any more in an analytical way. The increasing division of problems no longer functions. One needs a systematic approach and not a model in which only one variable changes. The planning system becomes the more chaotic, the more the work is being fragmented[39].

Today the weaker the students the stronger the fragmentation in education. The highly able get survey tasks as a report and a thesis.

1.4. Wide internal differences

Remedy **Open tasks**, which can be made in more ways and on more levels. 'Open' implies less steering. The student must depend on his/her own capacity, find his/her own way, must search for his/her own solution. This requires a different attitude from the teachers, who often are too quickly inclined to take away every obstacle.

A lot of investigations shows that as a general rule teachers, out of over-concern, remove obstacles as much as possible in learning. This occurs by dicing the learning matter into small problems, which are scarcely recognizable as parts of a whole. The activities are continually divided into sub-activities which should lead to routines and later are to be composed into larger algorithms. In present education everything is focused on the action rules. Precisely on the acquiring of rules, particularly arithmetic rules, a lot of time is spent. When teachers are asked what comes first, rules of calculation or conceptions, the answers are usually univocal:

"First of all, and this is most important, the pupils have to learn

exactly what the several steps of the calculation are. It is only on this basis that they may perhaps attain deeper insights or even conceptual understanding"..."Especially with weaker pupils, it is necessary to clarify the rules very carefully and step by step and to repeat them always in great detail; only a total mastery of rules will perhaps later on lead to a justification or to a meaning of these rules "...". Only on the basis of a complete and clear definition of mathematical concepts and procedures does it make sense to approach the understanding of conceptual relationships in mathematics teaching"...[40]

1.5. Low motivation, no self-control, passive

Remedy **Economical learning**

Not doing what has to be done, but knowing and learning with insight.

Too often, students are busy, but they are not learning much. It is basically keeping busy. The less able, especially, have to use their time as efficiently as possible.

It is remarkable that precise mechanistic training leads to uneconomic working and appeals to a well organized human memory. The point is that to remember sequences of routines or algorithms, organizing means and structures are required.

1.6. Ego-oriented, less communicative, social-emotional shortcomings

Remedy **Citizenship** mainly based on self-thinking, but also on communication and cooperation, access and use of instruments (calculators, computers, functions, formulas, variables). Citizens **are** not instruments, but **use** instruments.

This idea is not easy to realize, as it depends upon the image of men being equal. Unfortunately we have become so used to inequality.

1.7. Linguistically deprived

Remedy **Normal language and picture language**

To a certain extent language is not avoided. One often has to discuss, formulate and write down something.

The tasks are often illustrated, not only are pictures easily understood, but by picture language, alternative solutions or activities can also be presented. Then their own solution is easier compared with the solution in picture language.

1.8. Little self-confidence

Remedy **Subjective mathematics**

The student has to conceive that he/she creates his/her own mathematics. Mathematics is relating, and the student disposes of mathematical relationships.

Through the ages the relation between theory and practice (T-P) was seen as a problem, though the terms as such did not exist in the present sense. Basically it is the relation between work-a-day and scientific knowledge. The point of view is often to be found in the terminology, for instance 'co-operation' in stead of 'relation' or the problem of 'alienation' of science from the world of life. Both idealists and empiricists see T-P as an alienation problem. The one applauds the alienation because in this manner the true world will be better understood, the other regrets it, because the real will not be found.

For Plato was the relation dialectics - mathematics a T-P problem, as mathematical objects took an intermediate position between perceptible things and pure ideas.

real thing < mathematical object < pure idea

A mathematical object required a perceptible object, on the other hand the pure ideas could exist without it.

Mathematical development was characterized through the continuous purification, in which some of its principles were considered as non- mathematical. I will not outline the whole history of applied mathematics but just establish that in the nineteenth century applied mathematics no longer belonged to mathematics proper[41]. In fact, now we have a Platonian situation:

reality < applied mathematics < pure mathematics

Pure mathematics does not need real stimuli and applied mathematics is an interdisciplinary activity between mathematics and other qualifying disciplines.

Applying is most briefly expressed through looking for the general in the particular, i.e. whether the theoretical system can be interpreted in the real world. Essentially a correspondence is sought between an abstract, developed system often even completely free of perception a system of propositions and concepts, bonds and things of the world of experience. Often it is thought that theory is a luxury which the less able cannot afford, and applied mathematics could suffice. This view is purely objective and of an empiric kind.

In my contribution at Lochem[42] I explained that there is not an absolute separation between theory and practice as theory is continuously absorbed in reality. This would be nonsense when reality is considered objectively. Indeed there exists an objective reality, the subject of ontological investigations. But usually reality is seen as practice, as the **subjective real**, which is precisely our own and so not strange. Therefore mathematics is coming to one's own. To a mathematician, for instance, a complex number is familiar, thus real, like maybe to a teacher Piaget's cognitive structure. Similarly mathematical instruments have to be familiar and real for the less able.

1.9. Small basis of direction, restricted experience, mixed thinking forms with preference for informal thinking

Remedy **Elementary complementarism** of structure and application. Every task has in theory a structure aspect and an experience aspect.

Within this framework, the research about which Paeter Tulviste reported is interesting. It deals with the educational effect on syllogistic thinking. According to Wygotski the higher formal thinking, by symbol mediated human psychological functions, is changing by culture and not by nature. This thesis has been tested again on the Vai people in Liberia and on the Nganassan children of the Taimyr Peninsula[43]. The presented question was: 'Two people always drink tea together. Now the one is drinking tea. Is the other also drinking tea now?' The answer is fixed by the task.

Luria has already described that many people are supplying the context. Then one answers: 'no, they had a quarrel yesterday', or 'he could not come home, because of the awful weather'. School children on the contrary, says Tulviste, do not do this and are prepared to think from the given premise. Unfortunately the dates are missing, so one cannot control these statements. I doubt the result, and moreover, I would like to modify the proposition based on the pupils own experiences. At first the low attainers, especially, often mix thinking forms and are not willing to think scientifically. For instance 'rent a car'. Define from the data the most profitable offer, the Toyota or the Renault? Some pupils said: 'that's not necessary, because my father will never rent a Japanese car'. Furthermore, these children appear to think very quickly, they produce a lot of possible choices, but often they cannot concentrate sufficiently deeply and lose the overview. The tea-sample shows this magnificently. They do understand quite well what the logical implication should be: the other is drinking tea now too, but they immediately realize that one may not think inductively here. The past is in real life not the norm for the future. If you always got a high mark for your work that does not imply that you will get a high mark next time. Also in the tea-case, (s)he could be sick or even dead, or they had a quarrel. In fact these 'low attainers' are much cleverer than the syllogistic thinkers. It also explains why, after their school time, people still argue in their favour.

Here the higher form of abstraction is only a small aspect of the whole situation. It is required, but is not sufficient for understanding.

Some people argue that common-sense thinking suffices. This is a purely positivistic point of view in which structure is to be found in the context by reflecting.

It was a heavy blow for empiricism that mathematics appeared not to be a reflection of objective reality and, that qua objects, mathematics and reality differ essentially. This fundamental difference does not imply that the bonds between the real things and the relationships between mathematical objects also differ. One may even argue, for instance Poincaré did so, that outside the relations there is not a knowable reality. In any case mathematics is only to

be applied through these relations. The meaning of, let me say a number, never refers to a thing but to a relation between things.

So, the application of mathematics is always an active occurrence and especially from application it is understood that mathematics is an **activity** and not a bare system. With activity the mathematician came into the picture as the relating man and no longer as the perceiving man.

From time immemorial philosophers paid attention to the rules of mathematics, but one looked for the legitimation outside the knowing man - the subject - in the objective reason, in the dialectic, in spirit, god, etcetera. It was especially Kant who elaborated the turn to the subject, in conformity with the spirit of the time, by postulating the construction of knowledge and no knowledge as such. He was looking for the conditions of the construction. How does the mathematical system start and develop? Now his idea of mathematics as a synthesis apriori is well-known. Unfortunately it appears not tenable at first for geometry and the definitive turn from substantial thinking into relational thinking cleared the way for reasoning without the verifying perception. Earlier one thought of mathematics merely as an accurate description of real phenomena, now it appeared to be a system of pure creations of the mind, without any bearing on reality. It was this change of character which led to a schism between pure and applied mathematics i.e., the constitution of two different thinking forms: positivistic thinking and mathematical thinking. Hence with the definite turn from substance to subject, the old contradiction empiricism - idealism did not disappear. There were even neo-positivists like Reichenbach, who aimed at intuitive visualization, the only source from which a unification of science and the work-a-day world can spring[44]. For Reichenbach theoretical concepts would be superfluous if science would have been made intuitively.

In the pre-phase of modern mathematics objects led to relations, but the restarted mathematics works precisely in reverse, and so nowadays relations lead to general objects. In the abstract system in which the objects are known through the relationships, the development occurs under two contrasting methods or points of view:

A. The genetic or constructive method, illustrated by the

inductive definition of the natural numbers.

B. The axiomatic or postulational method, illustrated by the generalised Peano's axioms.

These methods emerge from the differences between empiricism and idealism. Is the result only defined by the axiomatic system, or by the general conditions of the construction? Is the mathematical theory transcendental or not? Are theory and practice of a different type?

The question is closely related to the image of man. The change into the 20th century was also the period of the foundational crisis, in which a mechanistic image of the world dominated. In this climate of cultural pessimism books such as Oswald Spengler's "Der Untergang des Abendslandes" appeared. Similarly the idea of mathematics as a deterministic system arose, with a fixed number of rules regulating the deduction of forms from forms. Just like a game of chess, H. Weyl said. Against this structuralistic pessimism, the optimism of existentialism broke through, stressing the independence of man and his freedom of thinking. There is no limit, even the set of the real numbers is non-enumerable as Cantor argued, one may always construct new ones. One can investigate infinite sets and their transfinite cardinals; even higher transfinite cardinals exist[45].

The fact is that knowledge as such sunk into the background and that since Kant the construction of knowledge is of primary importance. Mathematics as an activity has to be investigated. The formalist considers this activity in essence as deductive, while the constructivist thinks that activity is, to a certain extent, an inductive reasoning from which mathematics derives its fertility[46]. The formalist stresses activity as a structure, as the constructivist stresses the genese.

In both points of view one may indicate regularities, laws and patterns. With regard to the genese one discovered stadia for both the long historical process and the short process, i.e. the creative process. Structurally seen, the creative activity is not so well-known. Of course there are general statements as each activity is theoretically based and the hierarchy of logical types. Both are scarcely related to the process-control partition or simply the

how-why questions, by which is meant the how of the procedures and the why of the activity, hence the legitimations.

In spite of the definite turn of mathematics into a relating activity, based on free reasoning without the perception obstacle, there remains the substantial mathematics, the apparatus of enumeration and description of quantities. Substantial mathematics is the refined game of converting unities and multiplicities. It is purely positivistic thinking in which one must keep to perceptions and experience. It is especially this positivistic mathematics that carries through to the present day as arithmetics in our schools. The operations of addition and subtraction are obvious, multiplication and division are considered as repeated addition and subtraction and so on. Consequently one has two kinds of mathematics and mathematical thinking: the mathematics of common sense against the mathematics of intellect. The first deals with mental objects, to be found in history, the second with concepts defined in a system. Of course in history the concept follows the mental object. Unfortunately in epistemology there is no gradual change from substantial into relational. On the contrary there exists, what is called, an epistemological cut between both streams, particularly between the mental object and the mathematical concept.

What now? The common answer: 'Who cares, forget about relational mathematics, only substantial mathematics is necessary'. 'Forget about mathematical concepts, who needs groups, probability, functions, congruence and numbers as such?' 'Even people like Leibnitz worked with mental objects as the group'. Strange arguments, which with regard to biology, chemistry and physics imply that you should forget about modern insights, Leibnitz's level is enough. So much for mathematics! No, one has to deal with relational mathematics. But not in an empirical way.

In our everyday cognition, much remains at an elementary, low unconscious level. We eat a sandwich, spoon up our yoghurt, pay the gas bill, buy a train ticket, and go through many similar linear experiences, but usually linearity does not emerge.

Exponential experiences are still more rare, and are at a more submerged level. After a few steps, exponential series become more

and more complex. On each square of the chessboard grains are laid, on each successive square double the amount of the previous square. No one has a clear idea of what is on the fortieth square if one starts with one grain[47]. Only to a certain extent do we establish visual, audible and sensory relations, depending on the available means of language and mathematics. Thus we speak of: "best is cheapest", "the taller the thinner", "an harmonious orchestra" and "he only slowed down in the bends". For improving the description of our experiences we need an adequate mathematical language, but that is not enough. What a free man really needs is creative understanding and therefore a theory is necessary.

Undoubtedly human understanding started through comparing and through formulating and designing. Through reflecting we gained, for instance, insight into the linear relation. Spinoza explained the gradualness of this process in his "De verbetering van het verstand"[48],[49]. He called epistemology 'denckonst'. One does not need, he remarked, a second method to start the first one. As for forging an iron hammer you need a hammer. One begins, says Spinoza, with simple tools with which one makes more complicated tools, likewise:

> Reason creates by means of its innate power its own intellectual instruments, with which it creates new powers for new intellectual performances[50].

The innate power is not owned to external causes, says Spinoza. But in fact only a few people brought forward the entirety of knowledge and not humanity as a whole. The bright people did it, that is, they accelerated the process. Therefore the development process does not spontaneously start in the aforementioned cases of linear and exponentional relations. The biggest mistake we often make is following the historical process in education, thus following an empiricistic method, whether or not in an improved form. Experience is the less efficient way of learning.

The alternative is presenting a sufficient instrument to start with. As soon as it is presented the process of understanding can start, as well as understanding this instrument itself. Hence experiences and structures are compared, although an epistemological distinction exists between structure and experience. Man is able to connect

both parts metaphorically. Learning by new developed instruments is much more economical. Then the learner can been called the homo economicus. The entirety of instruments is a structure, but a mental structure, and not a reflection of reality, i.e. there is an epistemological distinction between structure and reality, although reality is better understood by means of the structure. The beginning, the foundation, thus the "innate power", is essentially a locking in. Man is also able to free himself from locking in, but it costs a lot of thinking energy.

It is of major concern that both thinking forms are involved. The instrument has to be developed within the theoretical thinking, by which confusion about the meaning is avoided. At the same time the theoretical instrument has to be applied. In this way a practical meaning is constituted. This will be illustrated in the next chapter.

1.10. Passive, less independent, no planning, no self-control

Remedy **Activity**, especially mental activity

> Answering the why and how of actions like counting, varying, measuring, playing and predicting. Dealing with the why and how means explaining, formulating, putting questions, designing and discussing, thus generally: being mathematically active.

Activity is not an easy conception. In dialectical materialism it is considered as a foundational concept between man and environment. Activity is the abstraction of human labour. In related educational circles the relation between man and fellow man is especially highlighted.

Thus human learning depends upon cooperation against the empiricistic idea that learning is the unmediated acquisition of objects. In the introduction of 'Learning and Teaching on a Scientific Basis'[31] the editors wrote:

> The concept of activity is the methodological key to understanding man as a societal being, not as an abstract individual.

The latter is an attack on structuralism.

Activity rises above actions, like procedures. Mathematics as an activity has nothing to do with real actions like cycling, running, cutting and sticking. Mathematics is only a mental activity and has

to do with mental procedures. Besides there are instruments for activity and products of activity, such as formulas, models, diagrams, equations and pocket-calculators. These instruments are mediating elements between theory and practice, between the why and the how.

But before looking at these, one has to observe a perhaps not so striking, but nonetheless very important point, namely the orientation of activity. One may go from the how to the why and reversely. In the first case one thinks positivistically and in the second idealistically. Consequently there are two kinds of activity or better kinds of thinking: positivistic thinking and - in our case - mathematical thinking. An opposition also to be observed in constructive thinking as opposed to postulational thinking. That implies that both the instruments and products are of different kinds too. An instrument is either a modelling representation of a real bond or an abstract relationship and a product can be a real change or a new theoretical element/part.

Theory regulates and controls its instruments, which in their turn regulate and create the procedures. Factually theory and practice are not directly connected, therefore mediating instruments are required. Besides, both orientations remain and so two loops exist. In one scheme of mathematics as an activity:

procedures ⟷ instruments ⟷ theoretical system

According to the way of thinking one has two different schemes:

real context ⟷ modelling representations ⟷ theoretical system

and

applications ⟷ relationships ⟷ theoretical system

It is often argued that real context and applications are to be identified. This is not true because not every context can be seen as an application. Anyhow the main point is the orientation of thinking; is it from T to P or from P to T? Realistic mathematics

education, as has been elaborated in my paper, 'The crux of realistic mathematics education', turns mostly into the first half of the scheme:

real context ⟷ modelling representations,

which is called horizontal mathematizing. The point is that from real contexts vertical mathematizing is almost impossible.

In short, positivistic thinking is constructive/inductive and is based on experience in the quality of real contexts, as mathematical thinking is existential/deductive and is based on relationships. Unfortunately both thinking forms lead to the ambivalence of instruments. An instrument seen as a model of a real situation is less general than as an abstract relationship. In the practice of education it leads to many misunderstandings as has been elaborated in my aforementioned paper.

Kant considered mathematics as a synthesis a priori, which appeared untenable. Mathematics is especially analytic, which implies that a form corresponds with some or many cases. Otherwise said: a form knows variables. For instance a complete quadrangle, consisting of four lines. Through choosing a pair of opposite sides a quadrangle is defined and so the four lines contain three quadrangles. A much easier example: a natural number is both a counting number and a measure number. On the contrary a model generated from a real bond corresponds only with one particular case. The cause is in the meaning which goes partly along with the representation.

Also in the case of mathematics education we may observe that instruments -learning materials- are used by the teacher in a manner not compatible with the theorist's intention. An example of this contradictory use can be found in the next chapter.

2. An example: The first question of 'Goede Maatjes'

Preliminary

Obviously one cannot begin with a very relative definition like 'measuring is finding the determinant of the gradient of a function'.

On the other hand saying that 'measuring is quantifying qualities' is also inscrutable. A more common and better answer is, that measuring is defining how many times a unit has been contained in a thing. But what is a unit? With 'a unit is a quantity with which one measures', one is in a cycle of definitions, resting on the substance-idea of mathematics. At present we know that measuring is the relation of mental objects. So, in order to measure real objects are not involved. Measuring time, measuring an area, measuring temperature, measuring weight, all are the measuring of objects that are not directly perceptible. The rotation of the earth, the falling of things, the melting of ice, and the surface of an object are to be perceived. But surface is not area, length is not long, time is not rotation. Only relating remains, it never concerns something as such but something in a relationship with something else. In short, measuring is a mental activity, by which something is thought of between two or more things. We can relate equal dimensions, as in 'velocity'. The relating of equal dimensions leads to partition-division and of unequal dimensions to proportion-division. Examined closely the difference does not exist, but for didactical purposes it is indeed useful. One may subtract equal dimensions and the partition-division corresponds then with the repeated subtracting of the same.

The first purpose
With measuring as a mathematical activity we meet the scheme:

procedures ⟷ instrument ⟷ theoretical system

As has been explained earlier the homo economicus is very apparent in the structure-experience connection. Therefore I did not want to start with measure procedures, but to deal with the whole scheme, extended to the relationship of length. In this case the Euclid algorithm is the instrument:

$$b = b_1a + a_1$$
$$a = b_2a_1 + a_2$$
$$a_1 = b_3a_2 + a_3$$
$$\ldots$$
$$a_{n-1} = b_{n+1}a_n + 0$$

for the commensurability of a and b.

N.B. a and b are numbers, and this algorithm is a model for measuring.

There are two main problems:

1. The meaning of the Euclid algorithm, that is, pacing out or measuring an object along another object, fixing the remainder, measuring this remainder along the first object, etc.
2. The metaphorical connection between the largest common divisor (l.c.d.) and the largest common unit (l.c.u.).

Hence I decided to proceed to the question about the metaphorical connection between the two questions about 'pacing out'. One called 'playing with mates' or 'play thirty-three' which is for two people.

One takes 33.

The beginner subtracts 3 or 5 and puts the remainder on paper.

The opponent subtracts also as desired 3 or 5 from the remainder and gives his remainder back to his partner and so on.

If at the end: the remainder is 1, then the beginner wins.
the remainder is 2, then the opponent wins.
the remainder is 3, then a remission.

One has to play it at least four times and then to answer the question: Can you always win? Then how? Write it down.

This is question two of the package.

Now something about the first problem: linear reduction

As previously mentioned there are two really different approaches concerning teaching:

P → G Start with particular and go to general, thus with **analogical thinking**. That is, present a number of concrete problems such as cutting off a thread, sawing off a plank, shovelling a heap of sand, shifting boxes, driving a car,

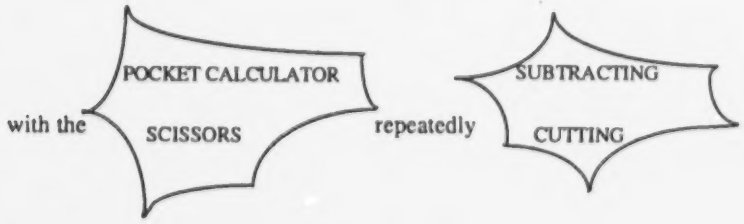
etc. Let the students solve these problems and let them reflect on these tasks in the expectation that abstractions are constituted. By mathematical formulation repeated subtracting will arise.

G → P Go **from general to particular**, so deal with a - bn, in which n is variable, then let the students work on various tasks, i.e. give a and b values, such as a=1000 en b=6, so 1000, 994, 988, 982, ... At the end let them search for concrete examples.

The first approach is rather **elitist** for it depends on experience. In many cases students cannot show experience hence the learning process is groundless. Furthermore it costs a lot of time to deal with all these experiences. Moreover the purpose is lacking. What is the sense of mathematizing?

The second approach easily becomes mechanistical. Pupils are active but without understanding. It is mainly a knowing in the activity in stead of a knowing of the activity. No wonder that applications appear to be extremely difficult and are mostly avoided by the teachers.

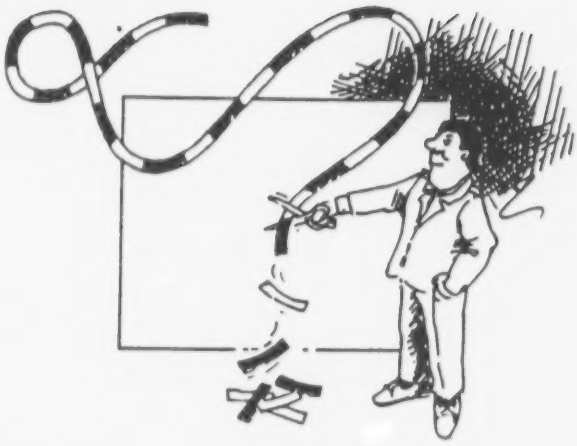
The third truly economical approach, elementary complementarism of structure and application, combines the profits of the foregoing ones. It rests on the hypothesis of two complexes, a structural one and a concrete one, plus the human metaphorical shifts within these complexes resting on only one connection. From this idea I formulated the following question:



Begin with 1000 and subtract repeatedly 16. Stop when you get a

number smaller than 16.

The REMAINDER is.....



The calculating machine is EXACT.
By the cutting off is there also exactly 8 left?
Yes, because .....
No, because .....

The points are:
- A repetition of repeated subtracting on the pocket-calculator, particularly the use of the repeat-key.
- The conception of the 'remainder'.
- The instrument 1000-16-16-16-...
- The metaphorical conversion of subtracting into cutting.
- An introduction of two thinking forms by means of the exactness of cutting.

The division 1000 : 16 must not be discussed. Naturally there are a lot of side-effects, like experiencing the size of 1000, and negative numbers. Most important is the idea of mathematics as a tool, making the real cutting action superfluous in order to find out the remainder and the idea of two thinking forms, a mathematical one and a practical/empirical one, standing side by side.

To avoid disorientation and to bring forward the real thinking,

education must deal with both sides: the mathematical activity and the 'corresponding' real activity, illustrated by subtracting and cutting off.

From a purely practical perspective the theoretical one is improbable or even improper. By isolating theory from practice one can freely be active in the theoretical field, but then it is blindly relying on activities with symbols. Here it is possible to test each singular aspect of theoretical thinking on its practical meaning. Everyone uses several thinking processes which function at the same time. One knows that mathematically 1000 = 62 x 16 + 8 and that physically it does not apply, because of the cutting faults which may be considerable. Anyway the latter can also be approached mathematically.

What about the equality of both activities? Pushing and cutting are real but different. Let me reverse the question. In what respect are both activities the same? The latter question involves the subject. Different realities can be mentally equal. The subject reasons, start with an extension and repeatedly take away a part. In other words pushing and cutting can be generalized into one real activity 'taking away'. This is pure inductive thinking with 'taking away' as the abstraction of subtracting and cutting. However subtracting was already abstract, so 'taking away' equals 'subtracting' or tape cutting equals mentally subtracting numbers.

But seen realistically the pairs (tape, cutting) and (number, subtracting) differ. Just look at the remainder: the one is eight, the other one will never be eight.

What does equality mean? Look at a=b. It is well known that there are two views: either by abstracting one gets new objects of higher complexity or one goes by concretizing from abstract objects to concrete objects. So they are either the same attributes of different substances or different forms of one substance.

Some experiences

The problem has been posed in several classes, even in two German classes. Of course I could not be present at all these places, consequently the following is a mixture of my own observations and the reactions of my colleagues.

The first part was generally not a serious problem. However not everyone knew of the repeat key. However without this key the question can still be answered. Every push has an immediate effect. In the long run the pupils do not see the effects any more and go too far. So, they discover in a natural way negative numbers and add 16, till they have got 8.

A report from a colleague:

At first the teacher has the pupil read the question: "First read the question well and then I will ask someone, what we have to do. I will not say who I shall ask." Having said this the task is obvious.

Two boys, neighbours, do it in their own way. Remarkably both choose a manner, behind which is a different thought:

1. Pocket-calculator; enter 16 + 16 = + 16 = ..., etc. This goes very quickly and being unaware 1024 is already in the display. "Oh, then it is 8, 24 - 16 = 8." The latter part is performed mentally!
2. "Now deducting 16's," ... calculating mentally ... "640 off" ... on the pocket-calculator 1000 - 640 = 360 ... "And now" ... again mentally "320 off" ... on the pocket-calculator 360 - 320 = 40 ... and further on the pocket-calculator: -16 = -16 = -16 = . On the display is now -8: "Oh, then the answer is 8."

Both methods well thought of...

A third boy finds an even different method:

3. Pocket-calculator: enter 1000-16 = -16 = -16 = ..., etc. Very soon appears in the display the sequence 24, 8, -8, -24, ... Oh, that sets him thinking! He himself tries it again, and again the same occurrence "Hey, he says to his neighbour and begins adding again, look what happens if you go on long enough", and he shows him his findings. The other is so busy with his own solution, that he scarcely looks up.

So much for the first part, now what is the similarity with cutting? Here one observes different experiences. One teacher reported that

the repeated subtracting and the cutting were not compared at first. He then took a rope of 1 m in length and performed the cutting procedure in front of the class. Other classes understood without the use of the rope demonstration. I saw also both phenomena in other classes.

Another teacher informed me that the cutting task was missing. He did not really understand the intention.

Obviously the exact-issue deepened the pupils insight. In some classes it led to discord: one group for exact, another against. That both points of view can be right is unsatisfactory.

Indeed the children show in almost every class metaphorical shifts both in the mathematical structure (e.g. 100 and 4) and in the concrete context (sawing a wooden plank, 1000 marbles).

It was reported that:

> On the question of the tape, almost everyone said no: "You cannot cut so exactly, you always cut a little slantingly" One girl said: "Yes, but you can work using a ruler". But nobody took up this suggestion. Everyone is indeed content with the answer, that you cannot work so exactly.

At the end some teachers 'misused' the problem. For instance in one German class, the teacher made a number of tapes of 10 meters in length. She gave each couple a tape and asked for the remainder if 16 cm was repeatedly cut off. She also said that later the question would be performed using a pocket calculator. She did not give the worksheet! Most pupils did not like it. What a task! One clever boy took a pocket calculator, used it and cut off 8 cm and said: this is the remainder!

Anyway in many cases it was observed that teachers did not seek certainty in mathematical instruments but in the reality of ropes, planks, marbles etc. *They did with this question what they pleased.*

NOTES

1. Terwel J., Onderwijs maken, The making of education, Stichting voor Onderzoek van het Onderwijs, 1984.

2. I quote his diary notes from 1850 (in my translation). Denken en zijn p.32, 1982, Boom, Meppel.

3. Freudenthal, H, Didactical Phenomenology of mathematical Structures, Reidel, Dordrecht, 1983.

4. Meant is his paper entitled: "An educational scientist in mathematics education: observer, teacher and theoretician". Later the book has been published: Steffe L.P., Von Glasenfeld, E., Richards J., Cobb P., Children's counting types: philosophy, theory and application, New York, Praeger, 1983.

5. Conference on Theory of Mathematics Education, held in Bielefeld.

6. "Translating mathematics learning research into practice" paper presented at the PME-conference in Bielefeld, July 1985.

7. Plato, The Republic, 509 d.

8. See for instance Habermans J. , Theorie und Praxis, Frankfurt am Main 1971, and Habermans J. - Luhman N., Theorie der Gesellschaft oder Sozialtechnologie, Frankfurt am Main, 1971, on a consensus - theory of truth.

9. Plato, The Republic 532a.

10. See for instance Halameisär and Seibt, N.I. Lobatschewski, Biographien, band 34, Teubner Verlag 1978, Page 63.

11. F. Engels: Anti-Duhring, 1878, XIII Dialectic.

12. The alternation theory-praxis resolves the dualism between the world in cosmological sense as the total of all phenomena and the world in transcendental sense. Kant used the term "world" next to the cosmological concept as the "unity of synthesis by imagination". An unity depending on schemata, regarded as time constituents a priori to rules (Kritik der reinen Vernunft, page 219, Von der Schematismus der reinen Verstandsbegriffe). Such schemata consequently lead to the many worlds of numbers, rules, potentials, etc. In present thinking it is common use to speak of many worlds based on distinctions in principles or activities like: the world of the book, the world of T.V., of science, of crime, of sport and not the least of mathematics education. If desired one can limit the above to the last one.

13. Die Erkenntnistheoretische Reflexie erweist sich als unersätzliches Mittel um aus den Grenzen wissenschaflicher Denkklischees auszubrechen und in neue Dimensionen der objektiven Realität vorzustoßen (Lektorski, Gesellschaftswissenschaften 1987-2 page 64).

14. Maybe the reader will argue that here a transposition of the praxis concept has occured for mathematics and physics differ from social sciences. Exact sciences do not show really the world, they do not understand for instance what temperature actually is. These theories are intentional closed. On the other hand the social sciences possess another truth concept. The logical power of the theory does not offer real truth, which rests on direct experience. This standpoint is feeble, both physic sciences and social sciences rest on the complementarity of structure ande experience, see for instance the work of W. Dilthey, E. Husserl on the one side and Popper, H. Albert on

the other.

15. Thinking about the comprehensive school (middenschool) in the Netherlands was largely an academic affair. Such brain children of educationists were tried out on a small scale but failed to bring praxis onto a higher level. Work in other schools remained virtually unchanged. Only one conclusion can be drawn: the theory was invalid, since there is no gap between theory and practice, there is only a praxis dissolving and at the same time raising theory.

16. The quotation is the opening sentence of W.R. Hamilton's Theory of Conjugate Functions, or Algebraic Couples from 1835. Quoted by M. Otte, What is a text, in Perspectives on Mathematics Education p.190.

17. Meant is: The dream of cooperation between theory and practice.

18. W. Quine also did not accept a strict distinction between theory and practice, which he particularly focused on perceptional and theoretical terms. The meaning of each term is also laid down by theory. It is about the theoretical-definiteness of perception, fierily defended by Kuhn and Feyerabend. The realists do not accept relativism, in other words not the context-definiteness of the truth of the statements.

19. Meant is: Theory and practice on paper and in action.

20. Recently I saw a relevant quotation of S. Boltzmann (in my translation):
there is nothing more practical than a good theory. But the only theory that is always applied immediately and directly is philosophical theory i.e. philosophy.
I cannot endorse just like that the really direct appeal of philosophy. Certainly it should be so, but many philosophers remain within the in-crowd and publish in specialist journals, which

are not generally accessible. Moreover, the common philosophical products are not understandable to everyone. I do concur with the rest of the quotation.

21. Karl Jaspers, 'Psychologie der Weltanschauungen' of 1922.

22. Look at: The dream of cooperation p.57.

23. Russell would probably use here the term another *logical* type.

24. The fourth international conference on the systematic relationship between theory and practice in mathematical education (SCTP) has as its theme 'the dialogue between theory and practice in mathematics education: overcoming the broadcast metaphor'.

25. Elaborated in: Theory and practice for the less able, p. 94 and further.

26. Activity has been already for some time the subject of a research programme which should lead to an activity theory. The theory should contain the complete explanation from the tendency to withdraw from the particularity of an empirically explained system. The main question is which place mas has in the material and idealistic reality of the activity theory. Besides that questions have to be answered like: is activity a fundamental principle upon which other forms of world appropriation may be constructed, so that activity can appear as a premise of the theory naturalistic ontology. In short, it is a world description reduced to an activity. Or, in case activity is not a fundamental principle: should it be explained as a epiphenomena with foundations in biological anthropology or historical sociology? The central question of methodology is the foundation of the activity theory. Does the theory describe or prescribe human conduct? In the latter case the activity theory is a philosophical conception of what people should actually do.

27. Hans Jörg Sandkühler formulated this during the 1st International Congress on Activity Theory, as follows:

> Die Tätigkeitstheorie stelt sich in eine Tradition der Geschichtsphilosophie, für die das Vico-Axiom repräsentativ ist: der Topos der geschichtlichen Selbsterzeugung des Menschen; Menschen können sich in ihrer Geschichte wiedererkennen und sich in Ihren Handlungen erklären, weil sie ihre Geschichte selbst machen. Doch scheint die Tätigkeitstheorie die geschichtsphilosophische und erkenntnistheoretische Weite des mit diesem Axiom begründeten Welt-, Selbst- und Wissenschaftsbildes zu schienen und das Feld möglicher Erklärung - statt es zu erweitern - eher einzuengen. Ein Indiz hierfür is die Häufigkeit der Konzeptualisierung von Tätigkeit durch eine bestimmte, den historischen Materialismus keineswegs in seiner Reichweite erschöpfende 'more oeconomico': Das Konzept Tätigkeit wird auffallend oft aus dem politisch-ökonomischen Begriff Arbeit abgeleitet, b.z.w. durch Analogie gewonnen. (Proceedings of the 1st International Congress on Activity Theory part 4.1.)

28. See e.g. L.S. Vygotsky, Thoughts and Language, M.I.T. Press, Cambridge, 1962.

29. Mellin-Olsen, The Politics of Mathematics Education, p. 151.

30. They have also different views on practice. SLO looks mainly at educational practice, the IDM especially at the practice of didactics.

31. See her paper 'Models of systematic cooperation between theory and practice' in Report of SCTP-2, ed. P. Verstappen, p.165, SLO, Enschede, 1988.

32. Weinzweig said for example: 'This learning and development for me is an ongoing process for I always learn from the teacher participants in the project.' in Report of SCTP-2, ed.

P. Verstappen, p.332, SLO, Enschede, 1988.

33. Report of SCTP-2, ed. P. Verstappen, p.254, SLO, Enschede, 1988.

34. Bernard Andelfinger saw a progress: learning psychology → pupils activities → human activities → constructive activities.

35. From this point of view it is probably useful to make a distinction according to the objects of theory, such as:
    - learning by children
    - teaching
    - the curriculum
    - curriculum development
    - the teacher
    - mathematics, e.g. number theory
    - teacher training
    - mathematics didactics.

36. for instance:
    One point of discussion was the philosophy of Zawadowski's innovational work. In the beginning he had no theory, only practice. Report of SCTP-2, ed. P. Verstappen, p.317, SLO, Enschede, 1988.

37. A first elaboration is to be found in for instance the Research Network Project, organized by the Consortium for Mathematics ans Science Education at the University at Albany, State University of New York, and supported by the Association of Mathematics Teachers of New York State and the Science Teachers Association of New York State. M.A. Farrell reported about it in her paper 'Research Practice Collaboration'.

38. That means something as: Looking at the development of mathematics for individual vocational training. 'OWI in de kijker' is an edition of the Institute for curriculum development SLO, september 1989, Enschede.

39. Nowadays one needs educated people, who can oversee the area. For more than this reason the unskilled workforce is confined to broad knowledge and capability.

40. H. Steinbring, Routine and meaning in the mathematics classroom. For the learning of Mathematics, 9.1, p.24-33.

41. An interesting paper on this subject appeared recently in 'Nieuw Archief voor Wiskunde', March 1990, H.M. Mulder.

42. The Dream of the cooperation between theory and practice.

43. In Gesellschaftswissenschaften 1989-2, Akademie der Wissenschaften der UDSSR, Moskou.

44. Hans Reichenbach, Selected Writings, Reidel, Dordrecht, p.316.

45. The logistic school (particularly Russell and Whitehead) rejected Cantor's idea of construction too, but for another reason. The research of paradoxes yielded the so called theory of types, in which the predicates have been ordered in a hierarchy of logical types. That is, a predicate regulates merely predicates of a lower level. Mixing up predicates from different logical types is not allowed. So, Cantor's diagonal construction of a real number does not yield a new number, since this number is defined unpredictably and thus belongs already to the given system.
    There have arisen three main schools of thought on the foundations of mathematics:
    1. the logistic school, mathematics is a branch of logic;
    2. the intuitionalistic school, mathematics is the human process of generation or construction;
    3. the formalistic school, mathematics is a system of consistent rules.

    Of course this is a rough partition, that can be refined by analyzing special topics like the infinite, the law of the exclu-

ded middle etc.. Besides, the first school appeared untenable as Gödel showed.

46. e.g. Poincaré, Science and hypotheses p.31.

47. Some insight may be gained by taking ten times as much instead of doubling and writing this down. On the fortieth square, forty digits must be written.

48. The improvement of reason.

49. Vertoog over de verbetering van het verstand, Spinoza, Korte geschriften, wereldbibliotheek, Amsterdam 1982, p. 437-495.

50. Schept het verstand door zijn aangeboren kracht zich zijn eigen verstandelijke werktuigen waarmee het nieuwe krachten schept tot nieuwe verstandelijke verrichtingen ...

51. Aarhus Universitet, Psykologisk Institut, 1984.